Class field theory, Hasse principles and Picard-Brauer duality for two-dimensional local rings

Takashi Suzuki

Chuo University, Japan

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1. Introduction

Notation:

- A: two-dimensional normal complete noetherian local ring with *perfect* residue field F
- K: its fraction field
- P: the set of height one primes ideals of A

What I talk today:

Construct class field theory and arithmetic duality for A (in the mixed characteristic case).

When the residue field F is finite, there is such a theory due to Saito:

- The abelian extensions of K, up to those completely split at all height one primes ideals of A, are classified by characters of the "K_2-idèle class group" of K.
- The completely split extensions are classified by the homology of the dual graph Γ of a resolution of singularities of A.
- There exists a "Hasse principle" exact sequence

$$0 \to H_1(\Gamma, \mathbb{Q}/\mathbb{Z}) \to H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \to \bigoplus_{\mathfrak{p} \in P} \mathbb{Q}/\mathbb{Z} \stackrel{\text{sum}}{\to} \mathbb{Q}/\mathbb{Z} \to 0.$$

- The Brauer group of K is Pontryagin dual to the (K_1-)idèle class group of K.

1.1. Background

On the other hand, there is Serre-Hazewinkel's geometric local class field theory:

Let k be a complete discrete valuation field with perfect residue field F such that $\mathrm{char}(F) > 0$. Then the group of units $\mathcal{O}_k^\times$ has a canonical structure as a perfect group scheme over F, which we denote by $\mathbf{O}_k^\times$ (perfect means having invertible Frobenius). For a perfect F-algebra R, we have

$$\mathbf{O}_k^\times(R) = \{\textstyle\sum_{n=0}^{\infty} \omega(a_n)\pi^n \mid a_n \in R,\ a_0 \in R^\times\},$$

where ω denotes the Teichmüller lift and π is a prime element of k. Let $\mathbf{k}^\times$ be the product of $\mathbf{O}_k^\times$ with the discrete group scheme $\pi^{\mathbb{Z}}$.

Theorem (Serre, Hazewinkel)

The abelian extensions of k are classified by isogenies onto $\mathbf{k}^\times$ with finite constant kernel.

1.2. The aim

Now we will refine Saito's two-dimensional theory for A in Serre-Hazewinkel's style:

- Put perfect group scheme structures on relevant arithmetic invariants.
- Allow the residue field F to be an arbitrary perfect field.
- Construct class field theory and arithmetic duality between these group scheme structures.
- We will focus on mixed characteristic A, though the equal characteristic case should also be interesting.
- Things are classical away from p $(= \mathrm{char}(F))$, so we will focus on the p-primary part.

2.1. Preliminaries: perfect group schemes

Let

- F: perfect field of characteristic $p > 0$
- Alg_u/F: the (abelian) category of perfections (inverse limit along Frobenius) of commutative unipotent algebraic groups over F
- For $G \in \mathrm{Alg}_u/F$, let G^0 be its identity component and $\pi_0(G) = G/G^0$ the (finite étale) group of components.

Serre duality

For any connected $G \in \mathrm{Alg}_u/F$, there exists a canonical connected group $G^{\mathrm{SD}} \in \mathrm{Alg}_u/F$ such that

$$G^{\mathrm{SD}}(F) = \varinjlim_n \mathrm{Ext}^1_{\mathrm{Alg}_u/F}(G, \mathbb{Z}/p^n\mathbb{Z})$$

and $(G^{\mathrm{SD}})^{\mathrm{SD}} \cong G$.

G^{SD} classifies isogenies onto G with finite constant kernel.

For example, for the perfection of the additive group $\mathbf{G}_a$, we have $\mathrm{Ext}^1_{\mathrm{Alg}_u/F}(\mathbf{G}_a, \mathbb{Z}/p\mathbb{Z}) \cong F$, where $1 \in F$ corresponds to the Artin-Schreier sequence $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbf{G}_a \to \mathbf{G}_a \to 0$. Correspondingly, $\mathbf{G}_a$ is self-dual, and the Frobenius automorphism on $\mathbf{G}_a$ corresponds to the inverse of Frobenius on $\mathbf{G}_a$. The group of Witt vectors W_n is also self-dual.

However, we need to treat "bigger" objects such as

$$W[1/p] = \varinjlim (W \xrightarrow{p} W \xrightarrow{p} \cdots) = \varinjlim_m \varprojlim_n W_n.$$

(Note that if $\mathcal{O}_k = F[[t]]$ so that $\mathcal{O}_k^\times \cong F^\times \times W(F)^{\mathbb{N}}$, then $\mathbf{O}_k^\times = \mathbf{G}_m \times W^{\mathbb{N}}$.)

2.1. Preliminaries: perfect group schemes

Definition

Define $\mathcal{W}_F$ to be the full subcategory of the ind-category of the pro-category of Alg_u/F consisting of objects G admitting a filtration $G \supset G^0 \supset G' \supset 0$ such that

- G/G^0 is finite étale,
- G^0/G' can be written as $\varinjlim_n G''_n$, where each $G''_n \in \mathrm{Alg}_u/F$ is connected and each $G''_n \to G''_{n+1}$ is injective, and
- G' can be written as $\varprojlim_n G'_n$, where each $G'_n \in \mathrm{Alg}_u/F$ is connected and each $G'_{n+1} \to G'_n$ is surjective with connected kernel.

So G^0 is build from $\mathbf{G}_a$ by a countable successive extension in an "ind-pro" manner. The subobject $G^0 \subset G$ is unique (but $G' \subset G$ is not), so we set $\pi_0(G) = G/G^0$.

2.1. Preliminaries: perfect group schemes

We say $G \in \mathcal{W}_F$ is connected if $G = G^0$.

Proposition

The Serre duality functor extends to connected groups in $\mathcal{W}_F$ in a way compatible with $\varinjlim$ and $\varprojlim$.

2.2. Preliminaries: cohomology groups of interest

Recall our notation:

- A: two-dimensional normal complete noetherian local ring with perfect residue field F such that $\mathrm{char}(F) = p > 0$
- K: its fraction field
- P: the set of height one primes ideals of A

Assume A has mixed characteristic.

- $X = \operatorname{Spec} A \setminus \{\text{the closed point}\}$
- $j\colon U \hookrightarrow X$: any dense open subscheme
- $H^q(U, \cdot\,)$: the étale cohomology functor

View X as an analogue of a *proper* smooth curve over a p-adic field.

2.2. Preliminaries: cohomology groups of interest

Define the compact support cohomology by
$H^q_c(U, \,\cdot\,) = H^q(X, j_! \,\cdot\,)$, where $j_!$ is the extension-by-zero functor.

For $r \geq 0$, let $\mathbb{Z}/p^n\mathbb{Z}(r)$ be the Bloch cycle complex mod p^n on U in the étale topology. For $r < 0$, let $\mathbb{Z}/p^n\mathbb{Z}(r)$ be the extension-by-zero of the usual Tate twist along $U \cap \operatorname{Spec} A[1/p] \hookrightarrow U$.

Then:

- $H^1(U, \mathbb{Z}/p^n\mathbb{Z})$ classifies abelian p-coverings of U.
- We have an exact sequence

  $$0 \to \operatorname{Pic}(U)/p^n \operatorname{Pic}(U) \to H^2(U, \mathbb{Z}/p^n\mathbb{Z}(1)) \to \operatorname{Br}(U)[p^n] \to 0.$$

- The group $H^3_c(U, \mathbb{Z}/p^n\mathbb{Z}(2))$ is the K_2-idèle class group mod p^n of U if $F = \overline{F}$ (will see this later).

Now we will put $H^q(U, \mathbb{Z}/p^n\mathbb{Z}(r))$ and $H^q_c(U, \mathbb{Z}/p^n\mathbb{Z}(r))$ algebraic structures over F and state a duality between them.

3.1. Main results: algebraic structures

Let $\overline{A}$ be the completed unramified extension of A and set $\overline{U} = U \times_A \overline{A}$.

Theorem (S.)

There exist canonical objects $\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$ and $\mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$ of $\mathcal{W}_F$ such that

$$\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))(\overline{F}) \cong H^q(\overline{U}, \mathbb{Z}/p^n\mathbb{Z}(r)),$$

$$\mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))(\overline{F}) \cong H_c^q(\overline{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$$

as $\mathrm{Gal}(\overline{F}/F)$-modules. These objects are zero unless $0 \leq q \leq 3$.

Theorem (S.)

There exist a Serre duality

$$\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))^0 \leftrightarrow \mathbf{H}_c^{4-q}(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2-r))^0$$

of connected groups in $\mathcal{W}_F$ and a Pontryagin duality

$$\pi_0(\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))) \leftrightarrow \pi_0(\mathbf{H}_c^{3-q}(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2-r)))$$

of finite étale groups over F.

3.3. Main results: finiteness

We will take a closer look into each cohomology object.

Theorem (S.)

The group $H^1(X, \mathbb{Z}/p^n\mathbb{Z})$ is finite if $F = \overline{F}$. In particular, the object $\mathbf{H}^1(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z})$ is finite étale (for any F).

In SGA 2, Grothendieck conjectures that $\pi_1(X)$ is topologically finitely generated. The above gives a weaker result.

3.3. Main results: finiteness

With the Serre duality

$$\mathbf{H}^3(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(2))^0 \leftrightarrow \mathbf{H}^1(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z})^0$$

and the Pontryagin duality

$$\pi_0(\mathbf{H}^3(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(2))) \leftrightarrow \pi_0(\mathbf{H}^0(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z})) \quad (\cong \mathbb{Z}/p^n\mathbb{Z}),$$

we have

Corollary

$$\mathbf{H}^3(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(2)) \cong \mathbb{Z}/p^n\mathbb{Z}.$$

3.3. Main results: finiteness

By Levine, the K_2-idèle class group of X is isomorphic to the Grothendieck group $K_0(\mathscr{C}_A)$ of the category of A-modules of finite length and finite projective dimension. The "length" map $K_0(\mathscr{C}_A) \to \mathbb{Z}$ is surjective if $F = \overline{F}$ by Levine. The previous corollary is equivalent to the following:

Corollary

The kernel of $K_0(\mathscr{C}_A) \twoheadrightarrow \mathbb{Z}$ is p-divisible if $F = \overline{F}$.

When A has equal characteristic, this divisibility is an unpublished result of Srinivas.

3.4. Main results: class field theory

The Pontryagin duality

$$\mathbf{H}^1(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}) \leftrightarrow \pi_0(\mathbf{H}^2(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(2)))$$

gives unramified class field theory for A.

The Serre duality

$$\mathbf{H}^1(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z})^0 \leftrightarrow \mathbf{H}^3_c(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2))^0$$

says that the abelian coverings of U "deformable to the trivial covering" are classified by isogenies onto the K_2-idèle class group of U.

3.5. Main results: Hasse principles

Assume $F = \overline{F}$ for simplicity. Let $H^1_{\mathrm{cs}}(X, \mathbb{Z}/p^n\mathbb{Z}) \subset H^1(X, \mathbb{Z}/p^n\mathbb{Z})$ be the subgroup consisting of coverings completely split at all $\mathfrak{p} \in P$. Set

$$\pi_0(\mathbf{H}^2(\mathbf{K}, \mathbb{Z}/p^n\mathbb{Z}(2))) := \varinjlim_{U \subset X} \pi_0(\mathbf{H}^2(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2))).$$

For any $\mathfrak{p} \in P$, the tame symbol $K_2(K) \to \kappa(\mathfrak{p})^\times$ to the residue field at $\mathfrak{p}$ followed by the valuation $\kappa(\mathfrak{p})^\times \twoheadrightarrow \mathbb{Z}$ defines a morphism

$$\pi_0(\mathbf{H}^2(\mathbf{K}, \mathbb{Z}/p^n\mathbb{Z}(2))) \to \mathbb{Z}/p^n\mathbb{Z}.$$

Let Y be the reduced part of the exceptional divisor of a resolution of singularities of A. Assume that Y is a strict normal crossing divisor. Let $Y_1, \ldots, Y_m$ be the irreducible components of Y. Let Γ be the dual graph of Y.

3.5. Main results: Hasse principles

(Still assuming $F = \overline{F}$)

Theorem (S.)

- The sequence

$$\pi_0(\mathbf{H}^2(\mathbf{K}, \mathbb{Z}/p^n\mathbb{Z}(2))) \to \bigoplus_{\mathfrak{p} \in P} \mathbb{Z}/p^n\mathbb{Z} \overset{\text{sum}}{\to} \mathbb{Z}/p^n\mathbb{Z} \to 0 \qquad (1)$$

  is exact.
- The kernel of the first map in (1) is Pontryagin dual to the group

$$H^1_{\mathrm{cs}}(X, \mathbb{Z}/p^n\mathbb{Z}) \cong H^1(Y, \mathbb{Z}/p^n\mathbb{Z}).$$

- We have an exact sequence

$$0 \to H^1(\Gamma, \mathbb{Z}/p^n\mathbb{Z}) \to H^1(Y, \mathbb{Z}/p^n\mathbb{Z}) \to \bigoplus_i H^1(Y_i, \mathbb{Z}/p^n\mathbb{Z}) \to 0.$$

3.6. Main results: Picard-Brauer duality

(Now, no need to assume $F = \overline{F}$)
Lipman defines a natural algebraic group structure (over F) on $\mathrm{Pic}(X)$ ($=$ the divisor class group of A). Let $\mathbf{Pic}_X$ be the perfection of this algebraic structure with identity component $\mathbf{Pic}_X^0$.

The group $\mathbf{Pic}_X^0$ surjects onto $\mathrm{Pic}_{Y/F}^0$ (which is a semi-abelian variety), and the kernel of this surjection has a filtration with graded pieces given by some coherent cohomology of Y (which is a vector group).

Let $\mathbf{Pic}_{X,\mathrm{sAb}}^0 \subset \mathbf{Pic}_X^0$ be the semi-abelian part and set $\mathbf{Pic}_X^0/\mathrm{sAb} = \mathbf{Pic}_X^0/\mathbf{Pic}_{X,\mathrm{sAb}}^0$.

3.6. Main results: Picard-Brauer duality

Define

$$\mathbf{Br}_X[p^\infty] := \varinjlim_n \mathbf{H}^2(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(1)).$$

Theorem (S.)

- The group of $\overline{F}$-points of $\mathbf{Br}_X[p^\infty]$ is $\mathrm{Br}(\overline{X})[p^\infty]$.
- We have $\mathbf{Br}_X[p^\infty]^0 \in \mathrm{Alg}_u/F$. The group $\pi_0(\mathbf{Br}_X[p^\infty])$ is cofinite (its part killed by p is finite étale).
- We have a Serre duality

$$\mathbf{Pic}_X^0/\mathrm{sAb} \leftrightarrow \mathbf{Br}_X[p^\infty]^0$$

of connected groups in Alg_u/F and a Pontryagin duality

$$T_p\mathbf{Pic}^0_{X,\mathrm{sAb}} \leftrightarrow \pi_0(\mathbf{Br}_X[p^\infty]),$$

where T_p denotes the p-adic Tate module.

4. Philosophical picture

A finitely generated module M over $W(F)$ naturally defines a perfect group scheme over F by $\mathbf{M} = M \otimes_{W(F)} W$. The object $\mathbb{Z}_p(r)$ over X should be identified with the mapping fiber of a divided Frobenius minus one on some prismatic cohomology. The prismatic cohomology should be a module over a big ring containing $W(F)$. Therefore, even if the divided Frobenius minus one is not linear over $W(F)$, it should at least be a morphism of group schemes over F, giving a group scheme structure on $H^q(X, \mathbb{Z}_p(r))$. That should agree with our algebraic structure $\mathbf{H}^q(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(r))$. Some duality for prismatic cohomology should induce a duality for $\mathbb{Z}_p(r)$-cohomology, which should agree with our duality results.

I do not know how much of this picture is actually realizable (note that A is not necessarily complete intersection). What I take is a more direct and traditional approach.

5.1. Ideas of proof: starting point

Key point of Saito's proof for finite F:

For a "nice enough" regular A, the groups $K_1(A[1/p])$ and $K_2(A[1/p])$ mod p have filtrations by symbols with graded pieces given by differential forms. On these graded pieces, the duality pairing is given by

$$F[[t]] \times (\Omega^1_{F((t))}/\Omega^1_{F[[t]]}) \stackrel{\text{residue}}{\longrightarrow} F \stackrel{\mathrm{Tr}_{F/\mathbb{F}_p}}{\to} \mathbb{Z}/p\mathbb{Z}$$

and similar pairings with $F[[t]]/F[[t]]^p$ or $F[[t]]^\times/F[[t]]^{\times p}$.

For general F:

- Replace $F[[t]]$ by the pro-algebraic group $\prod_{n\geq 0} \mathbf{G}_a t^n$ and $\Omega^1_{F((t))}/\Omega^1_{F[[t]]}$ by the ind-algebraic group $\bigoplus_{n\geq 1} \mathbf{G}_a t^{-n} dt$.
- Replace $\mathrm{Tr}_{F/\mathbb{F}_p}$ by the morphism $\mathbf{G}_a \to \mathbb{Z}/p\mathbb{Z}[1]$ in a derived category coming from the Artin-Schreier sequence $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbf{G}_a \to \mathbf{G}_a \to 0$.

5.1. Ideas of proof: starting point

For the whole groups (not just graded pieces) and bad A, it is not possible to just replace the groups "by hand". We need to work more functorially as follows.

For a perfect field extension F'/F (possibly transcendental), define the "base change of A to F'" by

$$\mathbf{A}(F') = \varprojlim_n (W(F') \otimes_{W(F)} A/\mathfrak{m}^n),$$

where $\mathfrak{m}$ is the maximal ideal of A. The ring $\mathbf{A}(F')$ is the same kind of object as A, except that the residue field is now F'. Treating $\mathbf{A}(F')$ in place of A, everything becomes a functor in F'. The group $F[[t]]$ in the previous page becomes the functor $F' \mapsto F'[[t]]$.

5.2. Functors in perfect field extensions

Do abelian groups functorially assigned to perfect field extensions F'/F uniquely pin down an object of $\mathcal{W}_F$? *Yes!*

A perfect artinian F-algebra is a finite product $F_1 \times \cdots \times F_m$, where each F_i is a perfect field extension of F. Note that an étale algebra over an perfect artinian F-algebra is again perfect artinian.

Definition

Define the *perfect artinian étale site* $\operatorname{Spec} F_{\mathrm{et}}^{\mathrm{perar}}$ to be the category of perfect artinian F-algebras endowed with the étale topology.

Let $\mathrm{Ab}(F_{\mathrm{et}}^{\mathrm{perar}})$ be the category of sheaves of abelian groups on $\operatorname{Spec} F_{\mathrm{et}}^{\mathrm{perar}}$.

Theorem

The natural Yoneda functor $\mathcal{W}_F \to \mathrm{Ab}(F_{\mathrm{et}}^{\mathrm{perar}})$ is fully faithful.

5.3. Cohomology as functors

Let $U \subset X$ be a dense open subscheme. For a perfect artinian F-algebra F', set

$$\mathbf{U}(F') = U \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbf{A}(F').$$

Now we define

$$\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)), \mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)) \in \mathrm{Ab}(F^{\mathrm{perar}}_{\mathrm{et}})$$

to be the étale sheafifications of the presheaves

$$F' \mapsto H^q(\mathbf{U}(F'), \mathbb{Z}/p^n\mathbb{Z}(r)),$$

$$F' \mapsto H_c^q(\mathbf{U}(F'), \mathbb{Z}/p^n\mathbb{Z}(r)),$$

respectively. There are derived categorical versions:

$$R\mathbf{\Gamma}(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)), R\mathbf{\Gamma}_c(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)) \in D(F^{\mathrm{perar}}_{\mathrm{et}}),$$

whose q-th cohomology objects are $\mathbf{H}^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$, $\mathbf{H}_c^q(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r))$, respectively.

5.4. Definition of the duality pairing

We define the duality pairing as follows:

We have a long exact sequence

$$\cdots \to H^q(X, \mathbb{Z}/p^n\mathbb{Z}(2)) \to H^q(K, \mathbb{Z}/p^n\mathbb{Z}(2)) \to \bigoplus_{\mathfrak{p}\in P} H^{q-1}(\kappa(\mathfrak{p}), \mathbb{Z}/p^n\mathbb{Z}(1)) \to \cdots .$$

If $F = \overline{F}$, then K has cohomological dimension 2 by Kato and $\kappa(\mathfrak{p})$ has cohomological dimension 1 by Lang. Hence $H^q(X, \mathbb{Z}/p^n\mathbb{Z}(2)) = 0$ for $q \geq 4$ and we have an exact sequence

$$K_2(K)/p^n K_2(K) \to \bigoplus_{\mathfrak{p}\in P} \kappa(\mathfrak{p})^\times / \kappa(\mathfrak{p})^{\times p^n} \to H^3(X, \mathbb{Z}/p^n\mathbb{Z}(2)) \to 0.$$

The sum of the valuation maps on $\kappa(\mathfrak{p})^\times$ is zero on $K_2(K)$ by the Quillen spectral sequence for A.

Therefore we have a map

$$H^3(X, \mathbb{Z}/p^n\mathbb{Z}(2)) \to \mathbb{Z}/p^n\mathbb{Z}$$

if $F = \overline{F}$. Hence for a general F and any algebraically closed F'/F, we have a map

$$H^3(\mathbf{X}(F'), \mathbb{Z}/p^n\mathbb{Z}(2)) \to \mathbb{Z}/p^n\mathbb{Z}$$

functorially in F'. This defines a morphism

$$\mathbf{H}^3(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(2)) \to \mathbb{Z}/p^n\mathbb{Z} \hookrightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

in $\mathrm{Ab}(F_{\mathrm{et}}^{\mathrm{perar}})$ and hence a morphism

$$R\mathbf{\Gamma}(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(2)) \to \mathbb{Q}_p/\mathbb{Z}_p[-3]$$

in $D(F_{\mathrm{et}}^{\mathrm{perar}})$, which is the *trace morphism* in this setting.

5.4. Definition of the duality pairing

With the cup product, we have morphisms

$$
\begin{aligned}
&R\mathbf{\Gamma}(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)) \otimes^L R\mathbf{\Gamma}_c(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2-r)) \\
&\to R\mathbf{\Gamma}_c(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2)) \\
&\to R\mathbf{\Gamma}(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(2)) \\
&\to \mathbb{Q}_p/\mathbb{Z}_p[-3].
\end{aligned}
$$

The composite

$$R\mathbf{\Gamma}(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)) \otimes^L R\mathbf{\Gamma}_c(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2-r)) \to \mathbb{Q}_p/\mathbb{Z}_p[-3]$$

is our duality pairing.

5.5. Duality theorem in sheaf form

For objects $C, D, E \in D(F^{\mathrm{perar}}_{\mathrm{et}})$, a morphism $C \otimes^L D \to E$ is said to be a *perfect pairing* if the induced morphisms

$$C \to R\,\mathbf{Hom}_{F^{\mathrm{perar}}_{\mathrm{et}}}(D, E) \quad \text{and} \quad D \to R\,\mathbf{Hom}_{F^{\mathrm{perar}}_{\mathrm{et}}}(C, E)$$

are isomorphisms, where $\mathbf{Hom}_{F^{\mathrm{perar}}_{\mathrm{et}}}$ denotes the sheaf-Hom functor for Spec $F^{\mathrm{perar}}_{\mathrm{et}}$.

Theorem

The morphism

$$R\boldsymbol{\Gamma}(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(r)) \otimes^L R\boldsymbol{\Gamma}_c(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}(2-r)) \to \mathbb{Q}_p/\mathbb{Z}_p[-3]$$

is a perfect pairing.

5.6. Proof sketch

Consider the following 3 statements:

1. this duality for $R\mathbf{\Gamma}, R\mathbf{\Gamma}_c$
2. the statement $\mathbf{H}^q, \mathbf{H}^q_c \in \mathcal{W}_F$
3. the duality for each $\mathbf{H}^q, \mathbf{H}^q_c \in \mathcal{W}_F$ (for π_0 and $(\,\cdot\,)^0$)

Then $(1) + (2) \implies (3)$.

To prove (1) and (2), the case where A is "nice enough" regular can be treated more or less by the same method as Saito's (by filtrations by symbols).

5.6. Proof sketch

To reduce the general case to the "nice enough" case, take a resolution of singularities $\mathfrak{X} \to \operatorname{Spec} A$ such that $\mathfrak{X} \times_A A/pA \subset \mathfrak{X}$ is supported on a strict normal crossing divisor. Consider the inclusions

$$X \overset{j}{\hookrightarrow} \mathfrak{X} \overset{i}{\hookleftarrow} Y,$$

where Y is the reduced part of the exceptional divisor. Using proper base change, write

$$R\mathbf{\Gamma}(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}(r)) \cong R\mathbf{\Gamma}(\mathbf{Y}, R\Psi\mathbb{Z}/p^n\mathbb{Z}(r)),$$

where $R\Psi = i^* Rj_*$ is the (p-adic) nearby cycle functor. Deal with the singularities of Y using the "nice enough" case, and then (essentially) give a duality for $R\Psi\mathbb{Z}/p^n\mathbb{Z}(r)$ and combine it with a p-adic duality theory for Y.

5.7. Finiteness of $H^1(X, \mathbb{Z}/p^n\mathbb{Z})$

($F = \overline{F}$) We may assume $n = 1$ and A contains a primitive p-th root of unity. Again, consider

$$X \overset{j}{\hookrightarrow} \mathfrak{X} \overset{i}{\hookleftarrow} Y.$$

We have an exact sequence

$$0 \to H^1(Y, \mathbb{Z}/p\mathbb{Z}) \to H^1(X, \mathbb{Z}/p\mathbb{Z}) \to \Gamma(Y, R^1\Psi\mathbb{Z}/p\mathbb{Z}) \to 0.$$

The term $H^1(Y, \mathbb{Z}/p\mathbb{Z})$ is finite. By Kummer theory, we may pass to (some part of) $\Gamma(Y, R^1\Psi\mathbb{Z}/p\mathbb{Z}(1))$. The sheaf $R^1\Psi\mathbb{Z}/p\mathbb{Z}(1)$ has a filtration by symbols with graded pieces given by coherent sheaves on Y. The negative-definiteness of the intersection pairing on $\mathfrak{X}$ gives some bound on the coherent cohomology. By some analysis of the Frobenius-fixed points of the coherent cohomology, we get the result.

5.8. Hasse principles

$(F = \overline{F})$ Let $U \subsetneq X$ be dense open. The long exact sequence for compact support cohomology gives an exact sequence

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \to \bigoplus_{\mathfrak{p} \in X \setminus U} \mathbb{Z}/p^n\mathbb{Z} \to \mathbf{H}^1_c(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z})$$
$$\to \mathbf{H}^1(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}) \to \bigoplus_{\mathfrak{p} \in X \setminus U} \mathbf{H}^1(\kappa(\mathfrak{p}), \mathbb{Z}/p^n\mathbb{Z}).$$

Taking the inverse limit in shrinking U, we get an exact sequence

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \to \prod_{\mathfrak{p} \in P} \mathbb{Z}/p^n\mathbb{Z} \to \varprojlim_{U} \mathbf{H}^1_c(\mathbf{U}, \mathbb{Z}/p^n\mathbb{Z}) \to \mathbf{H}^1_{\mathrm{cs}}(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z}) \to 0.$$

Taking the Pontryagin dual (noting that $\mathbf{H}^1_{\mathrm{cs}}(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z})$ is finite) and using our duality theorem, we get the desired exact sequence

$$0 \to \mathbf{H}^1_{\mathrm{cs}}(\mathbf{X}, \mathbb{Z}/p^n\mathbb{Z})^* \to \pi_0(\mathbf{H}^3(\mathbf{K}, \mathbb{Z}/p^n\mathbb{Z}(2))) \to \bigoplus_{\mathfrak{p} \in P} \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \to 0$$

(where $*$ denotes the Pontryagin dual).