# Harmonic analysis towards Manin-Peyre 

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## Sums of three cubes

Let $F_{0}=F_{0}(x, y, z):=x^{3}+y^{3}+z^{3}$. For each nonzero $a \in \mathbb{Z}$, the surface $F_{0}=a$ is known to have Zariski-dense sets of $\mathbb{Q}$-points [Segre 1943] and $\mathbb{Z}\left[a^{1 / 3}\right]$-points [Lehmer 1956, Beukers 1999, Hassett-Tschinkel 2001]. In general, producing $\mathbb{Z}$-points is harder, due to a lack of (known) structure.

Wooley proved $\left|F_{0}\left(\mathbb{Z}_{\geq 0}^{3}\right) \cap[0, A]\right| \gg A^{0.91709477}$ for $A \rightarrow \infty$. Conjecture (Deshouillers-Hennecart-Landreau): $F_{0}\left(\mathbb{Z}_{\geq 0}^{3}\right)$ has density $0.0999425 \ldots$ in $\mathbb{Z}_{\geq 0}$.

Classical notation for later (Hardy-Littlewood): Let

$$
r_{k}(a):=\#\left\{\left(x_{1}, \ldots, x_{k}\right) \in \mathbb{Z}_{\geq 0}^{k}: x_{1}^{k}+\cdots+x_{k}^{k}=a\right\}
$$

be the number of ways to write an integer $a \in \mathbb{Z}$ as a sum of $k$ nonnegative integer $k$ th powers.

## Sums of 3 cubes (cf. BSD; but less structure?)

Mordell '53:

- Maybe producing large, general ${ }^{1}$ integer solutions to

$$
x^{3}+y^{3}+z^{3}=a
$$

is as hard as "finding when an assigned sequence, e.g. 123456789, occurs in the decimal expansion of $\pi$ " ?

- Is there a solution for $a=3$ after

$$
3=1^{3}+1^{3}+1^{3}=4^{3}+4^{3}+(-5)^{3} ?
$$

In general, if solutions exist, they are expected to be very rare. ${ }^{2}$

[^0]
## The story of 33

Via computer, Booker obtained (at "five past nine in the morning on the 27th of February 2019")

$$
\begin{aligned}
(8866128975287528)^{3} & +(-8778405442862239)^{3} \\
& +(-2736111468807040)^{3}=33 .
\end{aligned}
$$

Later with Sutherland (September 2019):

$$
\begin{aligned}
(-80538738812075974)^{3} & +(80435758145817515)^{3} \\
& +(12602123297335631)^{3}=42 .
\end{aligned}
$$

Also,

$$
\begin{aligned}
(569936821221962380720)^{3} & +(-569936821113563493509)^{3} \\
& +(-472715493453327032)^{3}=3,
\end{aligned}
$$

thus affirmatively answering a question of Mordell.

## Hasse principle, BMOs, and beyond

[Heath-Brown 1992] has conjectured a quantitative Hasse principle: $x^{3}+y^{3}+z^{3}=a$ should have infinitely many solutions $(x, y, z) \in \mathbb{Z}^{3}$ for any fixed $a \not \equiv \pm 4 \bmod 9$.

For similar ( $\log \mathrm{K} 3$ ) surfaces, the Hasse principle can fail due to Brauer-Manin obstructions (BMO), ${ }^{3}$ or for more sophisticated reasons (e.g. combine BMO with group descent). ${ }^{4}$

Open Problem: Find new obstructions to the Hasse principle, for $\log$ K3 surfaces. Numerically [Ghosh-Sarnak 2022], there should be more obstructions for $x^{2}+y^{2}+z^{2}-x y z=a$ (Markoff-type surfaces) that we don't know yet.

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\({ }^{3} 5 x^{3}+12 y^{3}+9 z^{3}=a\) [Cassels-Guy 1966]
\({ }^{4} x^{2}+y^{2}+z^{2}-x y z=a\) [Ghosh-Sarnak 2022, Colliot-Thélène-Wei-Xu
2020, Loughran-Mitankin 2021]
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## Statistically producing sums of cubes

The fibers of the map $F_{0}:(x, y, z) \mapsto x^{3}+y^{3}+z^{3}$ are easier to understand on average than individually. ${ }^{5}$

The image $F_{0}\left(\mathbb{Z}^{3}\right)$ will be large if $F_{0}$ is nearly injective (in $\ell^{2}$ ). Note $\sum_{a \leq X^{3}} r_{3}(a) \sim C_{3} X^{3}$, and

$$
\sum_{a \leq X^{3}} r_{3}(a)^{2}=\#\left\{x \in \mathbb{Z}^{6} \cap X \cdot K: x_{1}^{3}+\cdots+x_{6}^{3}=0\right\}
$$

for some fixed compact region $K \subseteq \mathbb{R}^{6}$.
By Cauchy-Schwarz: The lower density of $F_{0}\left(\mathbb{Z}_{\geq 0}^{3}\right)$ is positive if $\#\left\{(\boldsymbol{y}, \boldsymbol{z}) \in[0, X]^{6}: F_{0}(\boldsymbol{y})=F_{0}(z)\right\} \ll X^{3}$ as $\bar{X} \rightarrow \infty$.
${ }^{5} \mathrm{Ex}: r_{3}(a) \gg a^{1 / 12}$ infinitely often (Mahler '36), but on average $r_{3}(a) \ll a^{\epsilon}$. Bounding $r_{3}(a), r_{4}(a), \ldots$ is an interesting problem (see Hypothesis K; cf. the $\ell$-torsion conjecture for class groups).

Now focus on $F(x)=F\left(x_{1}, \ldots, x_{6}\right):=x_{1}^{3}+\cdots+x_{6}^{3}$.
Definition
Let $N_{F, K}(X):=\#\left\{x \in \mathbb{Z}^{6} \cap X K: F(x)=0\right\}$, for $K$ a nice ${ }^{a}$ compact region in $\mathbb{R}^{6}$. (Or just use smooth weights!)

$$
{ }^{a} \text { Assume the boundary of } K \text { is suitably transverse to } F=0 \text {. }
$$

## Definition

Hardy-Littlewood ("randomness model") prediction for $F=0$ :

$$
N_{F, K}(X) \approx c_{H L} \cdot X^{6-3},{ }^{a}
$$

where the constant $c_{\mathrm{HL}}:=\sigma_{\mathbb{R}} \cdot \prod_{\rho} \sigma_{p} \in[0, \infty]$ is a product of local densities measuring the "local" (i.e. real and $p$-adic) bias of the equation $F=0$ (over the regions $K \subseteq \mathbb{R}^{6}$ and $\mathbb{Z}_{p}^{6}$ ).

[^1]
## Randomness and structure (for $F:=x_{1}^{3}+\cdots+x_{6}^{3}$ )

Hooley '86a: HL ("randomness") prediction misses trivial solutions (" $x_{i}+x_{j}=0$ in pairs"); maybe the truth is HLH?

## Conjecture ( HLH )

$N_{F, K}(X)=c_{H L} \cdot X^{3}+\#\left\{\right.$ trivial $\left.x \in \mathbb{Z}^{6} \cap X K\right\}+o\left(X^{3}\right)$ holds as $X \rightarrow \infty$.

## Remark (Around the square-root barrier)

1. The full HLH lies beyond the classical o-method (according to square-root "pointwise" minor arc considerations).
2. But the $\delta$-method ${ }^{a}$ opens the door to progress on HLH, by harmonically decomposing the true minor arc contribution in a "dual" fashion.
[^2]
## What's known towards HLH?

1. Hua '38: $N_{F, K}(X) \ll X^{7 / 2+\epsilon}$ (by Cauchy b/w structure and randomness in 4, 8 vars, resp.).
2. Vaughan ' $86+: N_{F, K}(X) \ll X^{7 / 2}(\log X)^{\epsilon-5 / 2}$ (by new source of randomness).
3. Hooley '86+: $N_{F, K}(X) \ll X^{3+\epsilon}$, under "Hypothesis HW" ( $\approx$ "modularity plus GRH") for the Hasse-Weil $L$-functions $L\left(s, V_{c}\right)$ associated to $V_{c}: F(x)=\boldsymbol{c} \cdot \boldsymbol{x}=0$.

## Remark

1. Hooley used an "upper-bound precursor" to the $\delta$-method.
2. The building blocks of the $\delta$-method are certain Fourier transforms.
We will see two kinds of behavior within: constructive (bias) and destructive (cancellation).

## Overview of Hooley's original approach

- Hooley's work uses the circle method (studying Fourier series in arcs $\left|\alpha-\frac{a}{q}\right| \leq \frac{1}{q Q}$, for $q \leq Q \asymp X^{3 / 2}$ and $a \perp q$ ), plus a clever use of an idea ${ }^{6}$ of Kloosterman '26, to reduce the additive counting question $N_{F, K}(X) \leq$ ? (about $F=0$ ) to estimating a beautiful but complicated average over $c \ll X^{1 / 2}$ of multiplicative quantities to moduli $q \leq Q$.
- This led to the surprising appearance ${ }^{7}$ of $1 / L\left(s, V_{c}\right)$ over $c \ll X^{1 / 2}$, which can be bounded for $\Re(s)>1 / 2$ under standard NT hypotheses, e.g. modularity plus GRH.
- After a significant amount of work this leads (conditionally) to the near-optimal estimate $N_{F, K}(X) \ll_{\epsilon} X^{3+\epsilon}$. By my count, there are four or five different sources of epsilon!

[^3]
## The $\delta$-method

The point count $N_{F, K}(X)$ looks like

$$
\begin{equation*}
\sum_{\boldsymbol{c} \in \mathbb{Z}^{6}} \int_{t \in \mathbb{R}}(\text { decay factor }) \prod_{p \text { prime }}(p \text {-adic geometry }) d t \tag{1}
\end{equation*}
$$

where the $p$-adic geometry comes from the complete exponential sums

$$
S_{\boldsymbol{c}}(n):=\sum_{1 \leq a \leq n: a \perp n} \sum_{1 \leq x_{1}, \ldots, x_{6} \leq n} e^{2 \pi i(a F(x)+c \cdot x) / n}
$$

which behave differently for $p \nmid \Delta(c)$ (smooth geometry) and $p \mid \Delta(c)$ (singular geometry).
Remark
Here $\boldsymbol{c}=\mathbf{0}$ (in (1)) produces HL term ( $c_{\mathrm{HL}} \cdot X^{3}$ ) but not full HLH (missing \#\{trivial $\left.x \in \mathbb{Z}^{6} \cap X K\right\}$ ).

The $S_{c}(n)$ 's relate to $V_{c}=\left\{[x] \in \mathbb{P}^{5}: F(x)=c \cdot x=0\right\}$. Fact: $\exists$ disc poly $\Delta \in \mathbb{Z}[c]$ measuring singularities of $V_{c}$.

Lemma (Hooley)
If $\Delta(c) \neq 0$, then $\widetilde{S}_{c}(n):=n^{-7 / 2} S_{c}(n)$ look (to 1st order) like the coeffs $\mu_{c}(n)$ of $1 / L\left(s, V_{c}\right)$.

## Partial proof sketch.

Here $F$ is homog (and $a$ is summed), so $S_{c}(n)$ is multiplicative. Locally: If $p \nmid \Delta(c)$, then $\widetilde{S}_{c}(p)=\widetilde{E}_{c}(p)+O\left(p^{-1 / 2}\right)$, where $\widetilde{E}_{c}(p):=p^{-3 / 2}\left[\# V_{c}\left(\mathbb{F}_{p}\right)-\# \mathbb{P}^{3}\left(\mathbb{F}_{p}\right)\right]$. Now use LTF (Lefschetz trace formula) and smooth $p$-adic geometry (Deligne).

Thus, at least over $\Delta(c) \neq 0$, we can use GRH to get square-root cancellation over the modulus $n$ in the delta method (up to $\epsilon$ and some singular $p$-adic geometry).

## Theorem (Hooley '86+/Heath-Brown '98)

$N_{F, K}(X)<_{\epsilon} X^{3+\epsilon}$, under Hypo HW ( $\approx$ modularity $+G R H$ ) for $L\left(s, V_{c}\right)$ 's (over $\left.\Delta(c) \neq 0\right)$. ${ }^{\text {a }}$
${ }^{a}$ A large-sieve hypo would suffice (W.). It's open! But $\exists$ uncond. apps to $x^{2}+y^{3}+z^{3}$ (W., via Brüdern ' $91+$ Duke-Kowalski ' $00+$ Wiles et al).

There are several critical sources of $\epsilon$ in Hooley/Heath-Brown, including the locus $\Delta(c)=0$ we have not yet discussed.

## Theorem (W. '21; unconditional)

The main terms of HLH come from the locus $\Delta(c)=0$.

## Proof hint.

We shall soon see why this is plausible (failure of "naive generalization" of $\mathrm{GRH} / \mathbb{F}_{p}$ caused by special subvarieties).

## Bias over finite fields

## Theorem (W. '22)

The following are equivalent for a cubic threefold $X$ of the form $x_{1}^{3}+\cdots+x_{6}^{3}=c_{1} x_{1}+\cdots+c_{6} x_{6}=0$ over $\mathbb{F}_{p}$ for $p \gg 1:^{a}$

1. $X$ fails the "naive generalization" of $G R H / \mathbb{F}_{p}$.
2. $X_{\overline{\mathbb{F}}_{p}}$ contains a plane.
3. $X_{\mathbb{F}_{p}}$ contains a plane lying on the Fermat cubic fourfold $x_{1}^{3}+\cdots+x_{6}^{3}=0$.
4. $X_{\mathbb{F}_{p}}$ contains $x_{1}+x_{2}=x_{3}+x_{4}=x_{5}+x_{6}=0$ (up to Fermat symmetries).
5. $c_{1}^{3}-c_{2}^{3}=c_{3}^{3}-c_{4}^{3}=c_{5}^{3}-c_{6}^{3}=0$ (up to symmetry).
${ }^{\text {a }}$ These hyperplane sections arise naturally in the context of the Fourier transforms $S_{c}(p)=\sum_{1 \leq a \leq p-1} \sum_{1 \leq x_{1}, \ldots, x_{0} \leq p} e^{2 \pi i\left(a\left(x_{1}^{3}+\cdots+x_{6}^{3}\right)+c \cdot x\right) / p}$.

The previous dichotomy follows from the following subtler, more general dichotomy. ${ }^{8}$

## Theorem (W. '22)

For a cubic threefold $X \subseteq \mathbb{P}_{\mathbb{F}_{p}}^{4}$ of the form $C\left(x_{1}, \ldots, x_{5}\right)=0$ with at most isolated singularities, the following are equivalent:

1. $X$ fails the "naive generalization" of $G R H / \mathbb{F}_{p}$.
2. There exist quadratic forms $Q_{1}, Q_{2} \in \overline{\mathbb{F}}_{p}\left[x_{1}, \ldots, x_{5}\right]$ "essentially in 4 variables", a and a homogeneous polynomial $A \in \overline{\mathbb{F}}_{p}\left[x_{1}, \ldots, x_{5}\right]$, such that $A \cdot C \in\left(Q_{1}, Q_{2}\right)$ and $A \notin \sqrt{\left(Q_{1}, Q_{2}\right)} .{ }^{b}$
3. $X_{\mathbb{F}_{p}}$ contains a plane or a singular cubic scroll.
> ${ }^{\text {a i.e. }} Q_{1}, Q_{2}$ with a common nonzero singularity
> ${ }^{{ }^{b}}$ I think it might be possible to take $\operatorname{deg} A \leq 1$, but have not checked.

${ }^{8}$ Reduction: A calculation-a singularity analysis—involving, among other things, $3 \times 3$ Vandermonde determinants arising from diagonality.

## Remark

The proof of the "more general dichotomy" combines classical geometry (including work of del Pezzo et al.), on the one hand, with amplificatory base change via modern geometry (Katz, Skorobogatov, et al.), on the other.

I like the statement ${ }^{9}$ more than the proof (which relies on some not-very-robust situation-specific geometry).

## Question

Is there a more enlightening or more general proof? Can one avoid or minimize use of base change? Can one use auxiliary polynomials or other tools?
${ }^{9}$ which, to me, is suggestive as to what may be true more generally

The significance of the threefold dichotomy is twofold:

1. It reflects "cubics are more random than quadrics". It gives an explicit "codimension 3" bound on the "locus of failures" of the "naive generalization" of GRH $/ \mathbb{F}_{p} .{ }^{10}$

- One can also (Lindner '20 + Lefschetz pencil theory) give an explicit "codimension 2" bound in terms of iterated discriminants (cf. Bhargava '22).
- Or (probably; cf. Grimmelt-Sawin '21) an inexplicit "codimension 2 " bound via perversity machinery of Fouvry-Katz '01 for Fourier transforms.

2. The dichotomy implies that special subvarieties in HLH/Manin for the cubic fourfold $x_{1}^{3}+\cdots+x_{6}^{3}=0$ "remain special" for hyperplane sections modulo $p$.

- On $x_{1}^{4}+x_{2}^{4}+x_{3}^{4}=x_{4}^{4}+x_{5}^{4}+x_{6}^{4}$, does a similar story hold for (Wooley's favorite special subvariety?)

$$
x_{1}+x_{2}+x_{3}=x_{4}+x_{5}+x_{6}=\left(x_{1}^{2}+x_{2}^{2}+x_{3}^{2}\right)-\left(x_{4}^{2}+x_{5}^{2}+x_{6}^{2}\right)=0 ?
$$

${ }^{10}$ This is consistent with Deligne-Katz equidistribution involved in RMT-type prediction recipes.

## Controlling singular $p$-adic geometry on average

One of the $\epsilon$ 's in Hooley/Heath-Brown comes from rather large failures of square-root cancellation in singular $p$-adic factors. To control this on average, we need the following ingredient:

Conjecture (B3, roughly; "cf. Sarnak-Xue")
For some $\delta>0$ : Over $c \in[-Z, Z]^{6}$ with $\Delta(c) \neq 0$, the probability there exists an integer $n \leq Z^{3}$ such that $\left|S_{c}(n)\right|$ fails square-root cancellation by a factor of $\geq \lambda \cdot n^{1 / 2-\delta}$ is $O\left(\lambda^{-2}\right)$.

The finite-field dichotomy and other related ideas let us prove (B3) assuming the Square-free Sieve Conjecture:

Conjecture (SFSC, roughly)
Over $c \in[-Z, Z]^{6}$ with $\Delta(c) \neq 0$, the probability there exists a prime $p \geq P$ with $p^{2} \mid \Delta(c)$ is $O\left(P^{-\delta}\right)$, for some $\delta>0$.
(B3) would fail if we replaced $x_{1}^{3}+\cdots+x_{6}^{3}$ with $x_{1}^{2}+\cdots+x_{6}^{2}$.

## Theorem (W. '21; conditional)

Roughly: Under RMT-type predictions ${ }^{\text {a }}$ and (B3), the locus $\Delta(c) \neq 0$ in the $\delta$-method contributes $O\left(X^{3}\right)$; in fact, o $\left(X^{3}\right)$. ${ }^{2}$ We use the Ratios Conjectures of Conrey-Farmer-Zirnbauer '08.

## Proof hint.

Appropriately decompose $S_{c}(n)$ to isolate distinct ${ }^{a}$ behaviors. ${ }^{b}$ For $O\left(X^{3}\right)$, use Hölder appropriately between "good" and "bad" factors; some important ingredients are (B3) and (R2'). For $o\left(X^{3}\right)$, handle some ranges (namely those with large "error moduli") the same. Over what remains, decompose $\Sigma_{\Delta \neq 0}$ into "error-constant" pieces-based on $\Delta$-up to a small exceptional set constructed by algorithmic tree-like means. Then estimate these pieces via local calculations and Poisson summation.

[^4]
## A sample RMT-type ingredient

Over $\Delta(c) \neq 0$, the reciprocal $L$-functions $1 / L\left(s, V_{c}\right)$ are the main players. The Ratios Conjectures imply e.g. the following:
Conjecture (R2', roughly)
Let $\sigma>1 / 2$ and $1 \leq N \leq X^{3 / 2}$. If $s=\sigma+i t$, then

$$
\mathbb{E}_{\substack{c \lll X^{1 / 2}: \\ \Delta(c) \neq 0}}\left|\int_{\mathbb{R}} d t e^{s^{2}} N^{s} \cdot \frac{\zeta(2 s)^{-1} L(s+1 / 2, V)^{-1}}{L\left(s, V_{c}\right)}\right|^{2} \ll N .
$$

- The LHS is independent of $\sigma .{ }^{11}$
- There are no $\log N$ or $\log X$ factors on the RHS! ${ }^{12}$
- This is enough "RMT input" for $N_{F, K}(X) \ll X^{3}$.
${ }^{11}$ One could take $\sigma-\frac{1}{2} \asymp \frac{1}{\log X}$ to facilitate comparison with other work.
${ }^{12}$ At least up to mollification/integration, logs reflect "symmetry type" of a family. Our $L$-functions are expected to behave like the characteristic polynomials of $C \times C$ random orthogonal matrices with $C \ll \log X$.


## Remark

- Up to Cauchy-Schwarz over s (losing $\log X$ ? ), (R2') might be very similar to well-studied moments (cf. Sound '09 and Harper '13 on moments of zeta, and Bui-Florea-Keating '21 and Florea '21 on negative moments of $L$-functions).
- But because the integral is inside in the absolute value, (R2') is really a statement about log-free cancellation over $n \asymp N$ of the coefficients of $\frac{\zeta(2 s)^{-1} L(s+1 / 2, V)^{-1}}{L\left(s, V_{c}\right)}$. This resembles the (unconditional!) log-free bound

$$
\frac{1}{X} \sum_{\substack{m \asymp x_{:}^{2}=\\ \mu(m)^{2}=1}}\left|\sum_{n \asymp x} \tilde{\lambda}_{f}(n)\left(\frac{m}{n}\right)\right|^{2} \ll X
$$

of Xiannan Li regarding certain orthogonal families of quadratic twists (see (1.3) of arXiv:2208.07343v2).

## More on mean values (Cancellation over $c$ )

The Ratios Conjectures also predict the following for $\sigma>1 / 2:{ }^{13}$
Conjecture (R1, roughly)
Write $s=\sigma+i t$. For some $\delta>0$ (independent of $\sigma$ ),

$$
\mathbb{E}_{\substack{c \ll X^{1 / 2}: \\ \Delta(c) \neq 0}}[\frac{1}{L\left(s, V_{c}\right)}-\underbrace{\zeta(2 s) L(s+1 / 2, V)}_{\text {polar factors }} A_{F}(s)]<_{\sigma, t} X^{-\delta}
$$

for $X \geq 1$. Here $A_{F}(s) \ll 1$ for $\Re(s) \geq 1 / 2-\delta$.

## Remark

For $N_{F, K}(X) \ll X^{3}$, we only use (R2'). But for HLH, we need a "slight adelic perturbation" (RA1) of (R1).
${ }^{13} \mathrm{~A}$ soft asymptotic for $\sigma-\frac{1}{2} \asymp \frac{1}{\log X}$ should also suffice for soft HLH.

## Main result

Theorem (W. '21)
Roughly: Assume standard NT hypotheses on L-functions and "unlikely" divisors. Then $N_{F, K}(X) \ll X^{3}$, and in fact HLH Conj. holds for a large class of regions K. (Actual hypo's for former are cleaner than those for latter.)

More precisely, hypotheses are the following:

- $L\left(s, V_{c}\right), L\left(s, V_{c}, \bigwedge^{2}\right), L(s, V)$ (Hypo HW2 + Ratios Conj's, where (R2') suffices for $N_{F, K}(X) \ll X^{3}$ ),
- Square-free Sieve Conjecture for $\Delta(c)$, and
- "effective Krasner" ${ }^{14}$ if one wants a power saving in HLH.

These are all essentially hypotheses about the family of Hasse-Weil L-functions $L\left(s, V_{c}\right)$ over $c \ll X^{1 / 2}$.

14"effective version of Kisin's thesis"

## Glossary for hypo's

1. Hypo HW2: Similar in spirit to Hooley's Hypo HW.
2. Ratios Conj's: Give predictions of Random Matrix Theory (RMT) type for mean values of $1 / L\left(s, V_{c}\right)$ and $1 / L\left(s_{1}, V_{c}\right) L\left(s_{2}, V_{c}\right)$ over families of $c$ 's. ${ }^{15}$
3. "Effective Krasner": Need $L_{p}\left(s, V_{c}\right)$ to only depend on $c \bmod p \Delta(c)^{1000}$ (cf. Kisin's thesis, Local constancy in $p$-adic families of Galois representations).
4. SFSC: Need, for $Z \geq 1$ and $P \leq Z^{3 / 2}$, an upper bound of $O\left(Z^{6} P^{-\delta}\right)$ for

$$
\#\left\{c \in[-Z, Z]^{6}: \exists p \in[P, 2 P] \text { with } p^{2} \mid \Delta(c)\right\} .
$$

[^5]
## Application to representing integers and primes

Theorem (W. '21, roughly)
Assume the same hypotheses as before. Then $N_{F, K}(X) \ll X^{3}$ for a large class of regions K. In fact, one gets an asymptotic featuring a randomness-structure dichotomy. ${ }^{a}$ Consequently, $100 \%$ of integers $a \not \equiv \pm 4 \bmod 9$ are sums of three cubes. ${ }^{b}$

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\({ }^{a}\) cf. conjectures of Hooley, Manin, Vaughan-Wooley, Peyre, et al. \({ }^{b}\) This follows from "HLH for sufficiently many \(K\) " (Diaconu ' \(19+\epsilon\) ).
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## Theorem (expected, but open)

Assume roughly the same hypotheses as above. Then $100 \%$ of primes $p \not \equiv \pm 4 \bmod 9$ are sums of three cubes. ${ }^{\text {a }}$
${ }^{a}$ This follows from "HLH with a power saving for sufficiently many $K$, with small divisibility constraints $d \mid x_{1}^{3}+x_{2}^{3}+x_{3}^{3}, x_{4}^{3}+x_{5}^{3}+x_{6}^{3 "}$.

## Approach to primes

To capture primes one can apply the Selberg sieve to a certain "approximate variance" for sums of three cubes. What would the Selberg sieve give towards the following question?

## Question

Assuming precise asymptotic second moments for $r_{3}(a)$ over $\{a \leq A: a \equiv 0 \bmod d\}$ for $d \leq A^{\delta},{ }^{a}$ can one show for $A \geq 2$

$$
\sum_{p \leq A} r_{3}(p)^{2} \ll A / \log A ?
$$

${ }^{\text {a }}$ The expected main term for these second moments may not vary multiplicatively with $d$. This may or may not be a serious obstacle.

Here $r_{3}(a):=\#\left\{(x, y, z) \in \mathbb{Z}_{\geq 0}^{3}: x^{3}+y^{3}+z^{3}=a\right\}$.
(The Selberg sieve does easily give $\sum_{p \leq A} r_{3}(p) \ll A / \log A$.)

## Questions to explore

- Prove (R2'), at least up to logs, under GRH? Cf. Sound '09, Harper '13, Bui-Florea-Keating '21, and Florea '21.
- Function-field analogs (GRH is known; exist monodromy groups; but only know limited ranges of RMT conjectures). ${ }^{16}$
- Understand the "subtle AG error factors" better; try to handle some non-diagonal analogs of $x_{1}^{3}+\cdots+x_{6}^{3}=0$ ?
- $x y z=u v w$ : NT basically understood ("multiplicative" harmonic analysis). Here can one go from NT to RMT?
- Hypothesis K (sparsity) fails for $x^{3}+y^{3}+z^{3}=a$. What about Hypothesis K for $x^{4}+y^{4}+z^{4}+w^{4}=a$ ? Lots of $A G$ questions in this vein.

[^6]
## Questions to explore (cont'd)

- Counting on quartics or other varieties: Try to combine symmetry (dynamical ideas?) and the circle method? Already exist many works using only one or the other.
- Find other applications of the Ratios Conjectures? For example, some extensions of my work to problems involving primes, or $a^{2} \pm b^{4} \pm c^{4}$, should be possible.
- Another recent application of RMT predictions (specifically, the Pair Correlation Conjecture) is to the density of composite $2^{n}+5$ [Järviniemi-Teräväinen 2020].
- Turán showed that the twin prime asymptotic conjecture is equivalent to a statement about cancellation over zeros of Dirichlet $L$-functions. However, there is no known reduction of twin primes to an RMT-type conjecture. ${ }^{17}$
${ }^{17}$ See "On the twin-prime problem III" (1968); perhaps the issue is the inhomogeneity of the problem? See $\bar{\chi}(-d, k)$ terms in Theorem B.


## Counting with symmetry

More progress (or less conditional progress) is possible under favorable structure like symmetry. For example, let $G=\left\{\left[\begin{array}{ll}a & b \\ 0 & 1\end{array}\right]\right\} \subseteq \mathrm{GL}_{2}$ be the $a x+b$ group, viewed as an algebraic group over $\mathbb{Q}$. Explicitly, the group law on $(a, b),(u, v) \in G$ is

$$
\begin{equation*}
(a, b) \cdot(u, v)=(a u, a v+b) \tag{2}
\end{equation*}
$$

## Theorem (W. '23)

Manin's conjecture holds for sufficiently split smooth equivariant compactifications $X$ of $G$ over $\mathbb{Q}$.

This builds on adelic harmonic analysis of Tanimoto-Tschinkel, who decomposed a point count on $G(\mathbb{Q})$ into the form

$$
\begin{equation*}
\sum_{\alpha \in \mathbb{Q}} \int_{t \in \mathbb{R}}(\text { decay factor }) \prod_{p \text { prime }}(p \text {-adic geometry }) d t \tag{3}
\end{equation*}
$$

## $p$-adic integrals

We build on adelic harmonic analysis of Tanimoto-Tschinkel, who decomposed a point count on $G(\mathbb{Q})$ into the form

$$
\begin{equation*}
\sum_{\alpha \in \mathbb{Q}} \int_{t \in \mathbb{R}}(\text { decay factor }) \prod_{p \text { prime }}(p \text {-adic geometry }) d t \tag{4}
\end{equation*}
$$

The $p$-adic geometry is, this time, given by integrals like

$$
\begin{equation*}
\int_{G\left(\mathbb{Q}_{p}\right): \alpha a \in \mathbb{Z}_{p}} H_{p}(s, g)^{-1} e\left(-\alpha b \bmod \mathbb{Z}_{p}\right)|a|_{p}^{-i t} d g \tag{5}
\end{equation*}
$$

Both additive and multiplicative harmonics appear above; the sum over $\alpha \in \mathbb{Q}$ somehow reflects the non-abelian nature of $G$. The "central term" $\alpha=0$, like $\boldsymbol{c}=\mathbf{0}$ in the $\delta$-method, gives the main term in the Manin-Peyre conjecture.

## Special divisors

Write $D:=X \backslash G=\bigcup_{j \in J} D_{j}$, where the $D_{j}$ are irreducible over $\mathbb{Q}$. Roughly, Tanimoto-Tschinkel handled the case where

$$
\begin{equation*}
\operatorname{ord}_{D_{j}}(a)<0 \Rightarrow \operatorname{ord}_{D_{j}}(b)<\operatorname{ord}_{D_{j}}(a) . \tag{6}
\end{equation*}
$$

Condition (6) relates to positivity of $K_{X}^{-1}$. Similar conditions, with variables and degrees, are familiar in the circle method.
Proposition (W.)
Let $j \in J$ and $c \in \mathbb{Q}$. Then $\operatorname{ord}_{D_{j}}(b-c) \leq \operatorname{ord}_{D_{j}}(a)$.
Definition (W.)
Given $j \in J$, call $D_{j}$ special if $\max _{c \in \mathbb{Q}} \operatorname{ord}_{D_{j}}(b-c)=\operatorname{ord}_{D_{j}}(a)$.
When (6) fails we seem to need a new idea. Main culprit: pairs of special divisors $\left(D_{j}, D_{i}\right)$ with $\operatorname{ord}_{D_{j}}(a) \operatorname{ord}_{D_{i}}(a)<0$.

## Local calculations

Suppose there are $k \geq 0$ special divisors with $\operatorname{ord}_{D_{j}}(a)<0$, and $l \geq 0$ special divisors with $\operatorname{ord}_{D_{j}}(a)>0$. Then the main issue, after new leading-order "bias" computations (in the spirit of our $\Delta(\boldsymbol{c})=0$ analysis in the $\delta$-method) relying on a new $G$-related source of local coordinates and cancellation in $p$-adic integrals, is to appropriately bound multiple Dirichlet series like

$$
\sum_{\substack{\alpha=m_{1} \cdots m_{k} / n_{1} \cdots n_{l}:}} \frac{f(\alpha) e(c \alpha)}{m_{1}^{\beta_{1}} \cdots m_{k}^{\beta_{k}}} \prod_{1 \leq j \leq 1} \frac{e\left(-c_{j} \alpha \bmod \mathbb{Z}_{n_{j}}\right)}{n_{j}^{\gamma_{j}}}
$$

pairwise coprime $m_{1}, \ldots, m_{k}, n_{1}, \ldots, n_{l} \geq 1$
for some $c, c_{1}, \ldots, c_{l} \in \mathbb{Q}$ and a hybrid additive-multiplicative Fourier transform $f: \mathbb{R}_{>0} \rightarrow \mathbb{C}$ of $H_{\infty}(s, g)^{-1}$. This is handled by analytic number theory methods: additive reciprocity (CRT) and multivariate Weyl-type inequalities in some ranges; and dyadic upper-bound harmonic analysis in other ranges.

## Open problems

Obtain a full log-power asymptotic point count on $G(\mathbb{Q})$ ? Right now, we leave delicate secondary terms unresolved in general (when the configuration of special divisors on the boundary $D$ is sufficiently complicated).

Counting points on $X(\mathbb{Q})$ versus $G(\mathbb{Q})$ ? Right now, the boundary is not included in the count.

Classify equivariant compactifications of $G$ ? For singular del Pezzo surfaces $X$, this is done by [Derenthal-Loughran 2015].

Look at other solvable groups, like the Heisenberg group?
Understand, deeply, the formal similarity between the $\delta$-method and the Tanimoto-Tschinkel G-harmonic analysis?


[^0]:    ${ }^{1}$ say non-parametric
    ${ }^{2}$ Cf. Hypothesis K of Hardy-Littlewood '25 that $r_{3}(a) \leq C(\epsilon) a^{\epsilon}$ for $a \geq 1$; it is false, but certainly $\mathbb{E}_{1 \leq a \leq A}\left[r_{3}(a)\right] \sim C$.

[^1]:    ${ }^{a}$ the -3 indicating "how hard it is to satisfy a cubic equation"

[^2]:    ${ }^{a}$ Kloosterman '26, Duke-Friedlander-Iwaniec '93, Heath-Brown '96

[^3]:    ${ }^{6}$ Poisson summation and averaging over a
    ${ }^{7}$ up to subtle algebro-geometric "error factors" related to a polynomial $\Delta(\boldsymbol{c})$ measuring the extent to which $V_{c}$ is singular

[^4]:    ${ }^{\text {a }}$ distinct at least under current philosophy
    bRoughly: "L-approximations", "good errors", and "bad factors".

[^5]:    ${ }^{15}$ Conrey-Farmer-Zirnbauer '08 build on other historical works, such as Conrey-Farmer-Keating-Rubinstein-Snaith '05, which in turn build on predictions for L-zeros "in the bulk" of Montgomery-Dyson '70s and others, and "near 1/2" of Katz-Sarnak '90s.

[^6]:    ${ }^{16}$ Ongoing work with Browning-Glas. Note also homological stability progress (EVW to present, large $q$ limit vs fixed $q$ ).

