Seminar on Arithmetic Geometry and Algebraic Groups

NETS FOR FISHING ABSOLUTE GALOIS PRO-p GROUPS

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AGAG seminar

Profinite groups and pro-p groups

Henceforth we fix an arbitrary prime p (it could be $p = 2$ with further conditions...)

A *profinite group* G is a topological group which is, equivalently, ...

- compact, Hausdorff, s.t. 1 has a basis of open neighbourhoods $\mathcal{N}$ consisting of normal subgroups;
- the projective limit of finite groups, i.e., $G = \varprojlim_{i \in I} G_i$, with $|G_i| < \infty$

A pro-p group is a profinite group G which is a projective limit of finite p groups, or, equivalently, G/N is a finite p group for every $N \in \mathcal{N}$.

EXAMPLES

here $p^n \to 0$ as $n \to \infty$ $\Rightarrow$ "$p^\infty = 0$"

- $\mathbf{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \ldots \mid a_n \in \mathbf{Z}/p\} = \varprojlim_{n \geq 1} \mathbf{Z}/p^n$ is an additive free cyclic pro-p group
- $\boxed{1 + p\mathbf{Z}_p} = \{1 + p\lambda \mid \lambda \in \mathbf{Z}_p\} \subseteq \underline{\mathbf{Z}_p^\times}$ is a multiplicative free cyclic pro-p group

Galois groups and profinite groups

Galois groups are profinite groups:

$$\mathrm{Gal}(\mathbf{L}/\mathbf{K}) = \varprojlim_{[\tilde{\mathbf{K}}:\mathbf{K}]<\infty} \mathrm{Gal}(\mathbf{L}/\tilde{\mathbf{K}}).$$

Conversely, every profinite group occurs as $\mathrm{Gal}(\mathbf{L}/\mathbf{K})$ for some field extension $\mathbf{L}/\mathbf{K}$, but...

BIG QUESTION(S)

- Which profinite group G occur as the *absolute Galois group* $G_{\mathbf{K}} = \mathrm{Gal}(\bar{\mathbf{K}}_s/\mathbf{K})$ for some field $\mathbf{K}$?
- Whic pro-p group occurs as the *maximal pro-p Galois group* $G_{\mathbf{K}}(p) = \mathrm{Gal}(\mathbf{K}(p)/\mathbf{K})$ for some field $\mathbf{K}$ containing $\sqrt[p]{1}$?

mxl pro-p quotient of G_K =

↳ mxl pro-p ext of K

Note: if G is pro-p and $G \not\simeq G_{\mathbf{K}}(p)$ for any field containing $\sqrt[p]{1}$, then $G \not\simeq G_{\mathbf{K}'}$ for any field $\mathbf{K}'$.

Orientations of pro-p groups

Definition

An *oriented pro-p group* (G, θ) is a pro-p group G equipped with a homomorphism of pro-p groups $\theta\colon G \to 1 + p\mathbf{Z}_p$ called an *orientation* of G.

The G-module $\mathbf{Z}_p(\theta)$ is defined by $\mathbf{Z}_p(\theta) \overset{\simeq \mathbb{Z}_p}{=} \{\, v \mid v \in \mathbf{Z}_p \,\}$, $g.v = \theta(g) \cdot v$

Oriented pro-p groups were introduced by I. Efrat 25 years ago with the name "pro-p pairs".

EXAMPLES $\quad K \in \sqrt[p]{1}$

- The *p-cyclotomic character* $\boxed{\theta_{\mathbf{K}}\colon G_{\mathbf{K}}(p) \to 1 + p\mathbf{Z}_p}$ satisfies $g.\zeta = \zeta^{\theta_{\mathbf{K}}(g)}$ for every $\zeta \in \bar{\mathbf{K}}_s$ root of 1 of p-power order
- A *Demuškin group G* yields a canonical orientation $\theta_G\colon G \to 1 + p\mathbf{Z}_p$ (if $G = G_{\mathbf{K}}(p)$ with $\mathbf{K}$ local then $\boxed{\theta_G = \theta_{\mathbf{K}}}$)

↳ mxl pro-p Gal gps of local fields

Pro-p groups of elementary type

We can combine together oriented pro-p groups:

2 or. pro-p gps

- (G_1, θ_1), (G_2, θ_2)

$$G_1 * G_2 \xrightarrow{\theta_1 * \theta_2} 1 + p\mathbb{Z}_p$$

free pro-p product

- (G, θ), $A \simeq \mathbb{Z}_p$

$A \rtimes_\theta G$, $\forall a \in A, g \in G$

$$g a g^{-1} = a^{\theta(g)}$$

Galois theory

- $G_{\mathbb{K}_1}(p)$, $G_{\mathbb{K}_2}(p)$ $\exists \mathbb{K}$ s.t.

$$\Rightarrow G_{\mathbb{K}_1}(p) * G_{\mathbb{K}_2}(p) = G_{\mathbb{K}}(p)$$

- $G_{\mathbb{K}}(p)$ $\mathbb{L} = \mathbb{K}((X))$

$\mathrm{Im}(\theta_{\mathbb{K}}) \subseteq 1 + p\mathbb{Z}_p$

$$\Rightarrow \boxed{G_{\mathbb{L}}(p) = A \rtimes_{\theta_{\mathbb{K}}} G_{\mathbb{K}}(p)}$$

$A \simeq \mathbb{Z}_p$

The Elementary Type Conjecture

The smallest family of oriented pro-p group which can be obtained starting from $(\{1\}, \mathbf{1})$, all $(\mathbf{Z}_p, \theta)$ (with θ arbitrary), and from Demuškin pro-p groups, employing these two operations, are called pro-p groups *of elementary type*.

Conjecture (I. Efrat, '97)

Let $\mathbf{K}$ be a field containing $\sqrt[p]{1}$. If $G_{\mathbf{K}}(p)$ is finitely generated, then $(G_{\mathbf{K}}(p), \theta_{\mathbf{K}})$ is of elementary type.

This conjecture is open, and also we don't know whether any oriented pro-p group (in fact, any Demuškin group) occurs as $(G_{\mathbf{K}}(p), \theta_{\mathbf{K}})$. Still:

- If a property is suspected to hold for all $G_{\mathbf{K}}(p)$'s, then prove it for oriented pro-p groups of E.T. first!
- Few concrete examples of pro-p groups which are not $G_{\mathbf{K}}(p)$'s are known: find them among non-E.T. oriented pro-p groups.

$\mathbf{Z}/p$-cohomology of pro-p groups

Consider $\mathbf{Z}/p$ as a trivial mod. of a pro-p group G: we write $H^k(G) := H^k(G, \mathbf{Z}/p)$.

- $H^0(G) = \mathbf{Z}/p$ cicly c gp order p
- $H^1(G) = \mathrm{Hom}(G, \mathbf{Z}/p) = (G/\Phi(G))^*$ ($\mathbf{Z}/p$-dual)
- if $\{1\} \to R \to F \to G \to \{1\}$ is a minimal presentation of G then $H^1(G) \simeq H^1(F)$ and

$$\left(\frac{R}{R^p[R,F]}\right)^* \xrightarrow{\sim} H^2(G)$$

so a basis of $H^2(G)$ "gives" a minimal set of defining relations of G (modulo $R^p[R,F]$)

Nets for fishing $G_{\mathbf{K}}(p)$'s

Voevodski (2011)

= Bloch-Kato conjecture

After the proof of the Norm Residue Theorem, we have three "cohomological nets" for fishing $G_{\mathbf{K}}(p)$'s among pro-p groups:

(1) A pro-p group G has the *Bloch-Kato property of deg 2* if

$H^n(G)$ depends only on $H^1(G)$ & $H^2(G)$

$$H^1(\cancel{G}H) \times H^1(\cancel{G}H) \xrightarrow{\ \smile\ } H^2(\cancel{G}H)$$

cup-product (anti-comm)

$\forall\, H \leq_o G$

$\Rightarrow H^2(H) \twoheadleftarrow \Lambda_2(H^1(H))$

(2) An oriented pro-p group (G, θ) is *1-cyclotomic* if the natural map

$$H^1(\cancel{G}H, \mathbf{Z}_p(\theta)) \xrightarrow{\ \mathrm{mod}\ p\ } H^1(\cancel{G}H)$$

$\forall\, H \leq_o G$

is surjective (note: G acts trivially on $\mathbf{Z}_p(\theta)/p) \cong \mathbb{Z}/p$

Both properties may be translated into group-theoretical terms. By the Norm Residue Theorem, $G_{\mathbf{K}}(p)$'s satisfy both properties.

Nets for fishing $G_{\mathbb{K}}(p)$'s

(3) A pro-p group G has the *n-Massey vanishing property* for $n \geq 3$ if the n-fold Massey product

$$\langle \alpha_1, \ldots, \alpha_n \rangle \subseteq H^2(G)$$

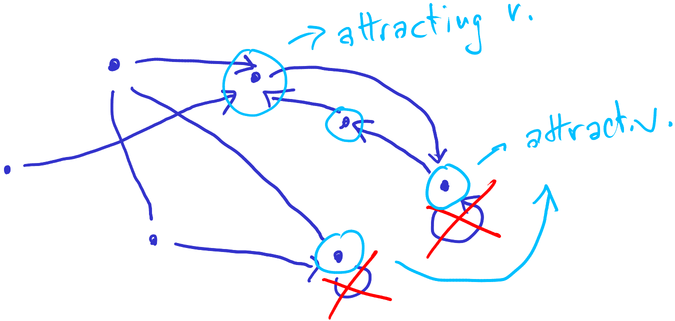
associated to an n-tuple of elements of $H^1(G)$ contains 0 whenever it is non-empty

$G_{\mathbb{K}}(p)$'s have the 3-MV property, and also the n-MV property for all $n \geq 3$ in some cases (number fields, local fields...), as well as oriented pro-p groups of E.T. It is conjectured that all $G_{\mathbb{K}}(p)$'s have the n-MV property for every $n \geq 3$.

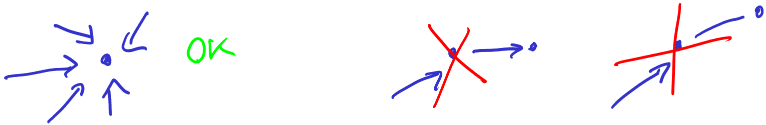
Mináč-Tân ('17)

Oriented graphs

My definition of *oriented graph* Γ: a finite set of vertices, some of which are joined by an arrow or by a plain edge — no loops allowed!



An oriented graph is *special* if every attracting vertex is a "black hole".



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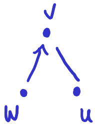
Oriented right-angled Artin pro-p groups

Given an oriented graph Γ, pick your favourite p-power p^f, $f \in \mathbf{N} \cup \{\infty\}$. The *oriented right-angled Artin pro-p group* associated to Γ and p^f is the pro-p group G with pro-p presentation

it could be $p^f = 0$ if $f = \infty$

$$G = \left\langle \text{vertices } v \;\middle|\; \begin{array}{cc} [v,w] = 1 & \text{if } v - w \\ vwv^{-1} = w^{1+p^f} & \text{if } w \to v \end{array} \right\rangle \hat{p}$$



with orientation $\theta_\Gamma \colon G \to 1 + p\mathbf{Z}_p$ defined by

$$\theta_\Gamma(v) = \begin{cases} 1 + p^f & \text{if } v \text{ attracting} \\ 1 & \text{if } v \text{ normal} \end{cases}$$

$$G = \left\langle u, v, w \;\middle|\; [u,v] = 1,\ v \cdot w \cdot v^{-1} = w^{1+p} \right\rangle$$

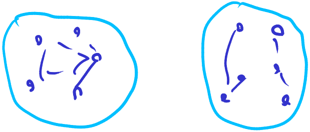
This is a very rich family of pro-p groups, so it is interesting from a Galois-theoretic point of view.

$\theta_\Gamma(v) = 1 + p^f$

$\theta_\Gamma(w), \theta_\Gamma(u) = 1$

Graphs of elementary type (starting from •)

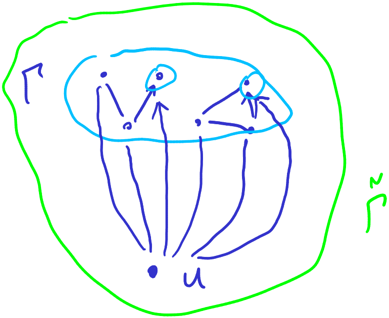
• $\Gamma_1 \,\dot\cup\, \Gamma_2$



• The or. pro-p RAAG ass. to $\Gamma_1 \,\dot\cup\, \Gamma_2$ is

$$G_1 * G_2, \quad G_i \text{ is ass. to } \Gamma_i$$

•



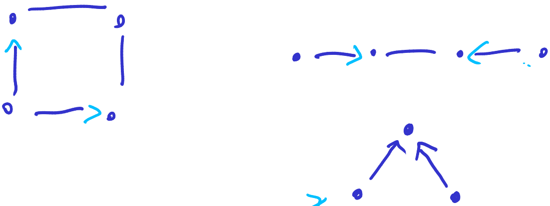
• The or pro-p RAAG ass to $\tilde\Gamma$ is

$$A \rtimes_{\theta_\Gamma} G \qquad G \text{ ass. to } \Gamma$$

$A \simeq \mathbb{Z}_p$

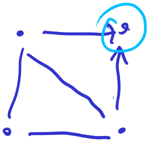
Graphs of elementary type

An oriented graph Γ is of E.T. if, and only if, Γ is special and it contains no induced subgraphs like



EXAMPLES:

NO



OK

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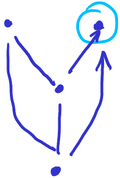

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Main Theorem

THM (Blumer-Weigel-Q)

Let Γ be an oriented graph, pick p^f, an let G be the associated oriented pro-p RAAG. TFAE:

(i) Γ is of E.T.

(ii) G is of E.T.

(iii) G has the BK prop. of deg. 2

(iv) G is 1-cyclotomic

(v) $G \simeq G_{\mathbf{K}}(p)$ for some $\mathbf{K}$ containing $\sqrt[p]{1}$ (in fact $\sqrt[p^f]{1}$)

This result...

- extends a recent result of Snopce-Zalesskiĭ (the same statement but without "oriented").
- provides a very large class of pro-p groups where the E.T. conjecture is verified
- provides a big wealth of concrete examples of pro-p groups (with a rather easy structure) which are not absolute Galois

Bad graphs 1

Bad graphs 2

Bad graphs 3

Oriented pro-p RAAGs and Massey products

On the other hand, also some bad oriented pro-p RAAGs have the n-Massey Vanishing property:

If Γ is a special oriented graph, then the oriented pro-p RAAG associated to Γ and any p^f has the n-M.V. property for every $n \geq 3$.

To prove this we use the following group theoretic characterization: a pro-p group G has the n-M.V. property if, and only if, ...

$$\begin{tikzcd} & & & G \arrow[dl, dashed, "\rho"'] \arrow[d, "\bar{\rho}"] & \\ \{1\} \arrow[r] & \mathrm{Z}(UT_n(\mathbf{Z}/p)) \arrow[r] & UT_n(\mathbf{Z}/p) \arrow[r] & \dfrac{UT_n(\mathbf{Z}/p)}{\mathrm{Z}(UT_n(\mathbf{Z}/p))} \arrow[r] & \{1\} \end{tikzcd}$$

Relations between nets

How are the 3 cohomological properties related?

Florence-De Clercq

• deg 2 BK $\overset{?}{\Longleftarrow}$ • 1-cyc.

WEAKER

• n-MV prop

References

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- I. Efrat, Orderings, valuations, and free products of Galois groups, *Sem. Struct. Algébriques Ordonnées Univ. Paris VII* (1995).
- C. Haesemeyer and Ch. Weibel, *The norm residue theorem in motivic cohomology*, Annals of Mathematics Studies, vol. 200, *P.U.P.*, 2019.
- C. Quadrelli and Th. Weigel, Profinite groups with a cyclotomic p-extension, *Docum. Math.* (2020).
- I. Snopce and P. Zalesskiĭ, Right-angled Artin pro-p groups, *Bull. L.M.S.*, in press.

Last slide

THANK YOU!