Milnor K-theory and zero-cycles over p-adic function fields

(j.w. w/ G. Lucchini Arteche)

Def (Artin–Lang): K field, $i \geq 0$.

K is C_i if $\forall X \subseteq \mathbb{P}^n_K$ hypersurface of degree d with $d^i \leq n$,
$X(K) \neq \emptyset$.

Ex.:
1) K C_0 $\Leftrightarrow$ $K = \bar{K}$. — $\mathrm{cd}\,\bar{K} = 0$.
2) Chevalley–Warning Th.: finite fields are C_1. — $\mathrm{cd}\,\mathbb{F} = 1$.
3) Tsen–Lang–Nagata Th.:
$$\left.\begin{array}{l} K \; C_i \\ \deg\mathrm{tr}(K'/K) = \delta \end{array}\right\} \Rightarrow K \; C_{i+\delta}.$$
$\mathrm{cd}\,K' \leq \mathrm{cd}\,K + \delta$.
4) Greenberg's Approximation Th.: K C_i $\Rightarrow$ $K((t))$ C_{i+1}. — $\mathrm{cd}\,K((t)) = \mathrm{cd}\,K + 1$.

Counter-ex.:
1) $K \subseteq \mathbb{R}$.
2) p-adic fields: Terjanian, Arkhipov–Karatsuba, Alemu: $\mathbb{Q}_p$ are not C_i.

Cohomology: $K \rightsquigarrow$ Galois cohomology.

Cohomological dimension:
$$\mathrm{cd}(K) = \max\{n \mid H^n(K, M) \neq 0 \text{ for some } \boxed{\text{finite}} \text{ Galois module } M\}.$$

Question: K C_i $\Leftrightarrow$ $\mathrm{cd}(K) \leq i$?

NO!!! : $\mathbb{Q}_p$ — not C_2; — $\mathrm{cd}\ 2$. ($\Leftarrow$).

($\Rightarrow$) Open: Question of Serre: known for
- $i=1$ easy.
- $i=2$ (Suslin).
- $i \geq 3$?

1) Milnor K-theory and Kato and Kuzumaki's conjectures.

Def.: K field, $q \geq 0$.
The q^{th} Milnor K-theory group of K is:

$$K_q(K) := \begin{cases} \mathbb{Z} & \text{if } q=0. \\ \underbrace{K^\times \otimes \cdots \otimes K^\times}_{q \text{ terms}} \Big/ \langle x_1 \otimes \cdots \otimes x_q \mid \exists i \neq j,\ x_i + x_j = 1 \rangle. \end{cases}$$

NORM: L/K finite extension:

$q=0$: $K_0(L) = \mathbb{Z} \xrightarrow{\times [L:K]} K_0(K) = \mathbb{Z}$.

$q=1$: $K_1(L) = L^\times \xrightarrow{N_{L/K}} K_1(K) = K^\times$

$q \geq 2$: $K_q(L) \xrightarrow{N_{L/K}} K_q(K)$ (Th. Kato).

Def. (Kato & Kuzumaki '86):

① $q \geq 0$. Z/K variety.

$$N_q(Z/K) := \left\langle N_{L/K}(K_q(L)) \;\middle|\; \begin{array}{l} L/K \text{ finite} \\ Z(L) \neq \phi \end{array} \right\rangle \subseteq K_q(K).$$

② $q, i \geq 0$. The field K is C_i^q if:
$\forall L/K$ finite, $\forall Z \subseteq \mathbb{P}^n_L$ hypersurface of degree d with $d^i \leq n$,

$$N_q(Z/L) = K_q(L).$$

Ex.: 1) $q=0$:

$$N_0(Z/K) = \langle [L:K] \mid Z(L) \neq \phi \rangle \subseteq K_0(K) = \mathbb{Z}.$$
$$= \operatorname{ind}(Z)\, \mathbb{Z}$$
$$\operatorname{ind}(Z) = \gcd([L:K] \mid Z(L) \neq \phi).$$

K C_i^0 $\iff$ $\forall L/K$ finite, $\forall Z \subseteq \mathbb{P}^n_L$ ---- $d^i \leq n$, $\operatorname{ind}(Z) = 1$. ($\iff Z$ has a 0-cycle of degree 1).

$\Uparrow$

K C_i

2) $i=0$: $Z = \operatorname{Spec} M$, M/L finite ext.

$$N_q(Z/L) = \operatorname{im}(N_{M/L} : K_q(M) \to K_q(L)).$$

K C_0^q $\iff$ $\forall M/L/K$, $N_{M/L} : K_q(M) \to K_q(L)$ surjective.

Prop (Kato & Kuzumaki '86, Bloch-Kato conj by Rost-Voevodsky):

K is $C_0^q \iff \mathrm{cd}\, K \le q$.

Conj. (KK '86): K is $C_i^q \iff \mathrm{cd}\, K \le q+i$.

$\boxed{C_i^0 \overset{?}{\iff} C_{i-1}^1 \overset{?}{\iff} \cdots \overset{?}{\iff} C_1^{i-1} \overset{?}{\iff} C_0^i \iff \mathrm{cd} \le i.}$

diophantine $\longleftrightarrow$ K-theory

Answer: NO: $\to$ Merkurjev '90: cd 2. $\cancel{C_2^0}$

$\to$ Colliot-Thélène / Madore '91: cd 1.

Constructed by transfinite induction.

Q. (Wittenberg '15): Can one prove the conjectures for fields appearing "naturally" in arithmetic geometry?

$\to$ Tot. imag. number fields & p-adic fields (coho. dim 2):

C_0^2 ✓ (KK).

C_1^1 ✓ (Wittenberg '15).

C_2^0 ?

$\to$ Function fields of complex varieties (– '17).

2) Function fields of p-adic curves.

$K = k(C)$: $\left|\begin{array}{l} k \ \ p\text{-adic} \\ C/k \ \ \text{smooth proj. geom. integral curve.} \end{array}\right.$

$\Rightarrow \mathrm{cd}\, K = 3$: expect C_i^q for $i+q \ge 3$

- $q \ge 3$: ✓ (KK).
- $\underline{\underline{q=2}}$ $\quad C_i^2$ for $i \ge 1$?

Th. A: l/k finite unramified extension, $L := l \cdot K$. Z/K proper variety. Then the quotient:
$$K_2(K) / \langle N_2(Z/K), N_{L/K}(K_2(L)) \rangle$$
is killed by $\chi_K(Z,E)^2$ for every coherent sheaf E over Z.
$$\chi_K(Z,E) = \sum (-1)^i \dim_K H^i(Z,E).$$

Cor.: K has property C_2^2. $d \leq n$

Pf.: $Z \subseteq \mathbb{P}^n_K$ deg d w/ $d^2 \leq n$.

Want: $N_2(Z/K) = K_2(K)$?

k^{nr} C_1 $\overset{\text{Tsen-Lang-Nag.}}{\Rightarrow}$ $k^{nr}((t))$ C_2. $k((t))$ C_2.

$\overset{d^2 \leq n}{\Rightarrow}$ $Z(k^{nr}((t))) \neq \emptyset$.

$\Rightarrow$ $\exists\, l/k$ ~~unramified~~ finite $\boxed{Z(l((t))) \neq \emptyset}$

$L := lK$: By Th. A:
$$K_2(K)/N_2(Z/K) = K_2(K)/\langle N_2(Z/K), N_{L/K}(K_2(L)) \rangle.$$
is killed by $\chi_K(Z, \mathcal{O}_Z)^2$.

But: $\chi_K(Z, \mathcal{O}_Z) = 1$. $\square$

Th. B: l/k finite, $L := lK$, Z/K proper.

$\exists\, s \geq 0$, $K_2(K)/\langle N_{L/K}(K_2(L)), N_2(Z/K) \rangle$ is killed by $i_{ram}(C) \cdot \chi_K(Z,E)^s$ for every coherent sheaf E over Z.
$$i_{ram}(C) := \gcd(\underbrace{e(k'/k)}_{\text{ramification degree}} \mid C(k') \neq \emptyset).$$

Cor.: $\forall\, Z \subseteq \mathbb{P}^n_K$ hypersurface of deg $d \leq n$,
$$K_2(K)/N_2(Z/K)$$
is killed by $i_{ram}(C)$. In particular, $K_2(K) = N_2(Z/K)$ if $i_{ram}(C) = 1$. (in particular if $C(k) \neq \emptyset$).

3) Idea of Proofs.

Th. A: $x \in K_2(K)$.

Want: $\chi_K(Z,E)^2 \cdot x \in \langle N_2(Z/K), N_{L/K}(K_2(L)) \rangle$.

Step 3: Solve the problem locally: for v a point in C.

Find $M_i^{(v)}/K_v$ $(1 \le i \le r_v)$ s.t.:

$$\chi_K(Z,E)^2 \cdot x \in \langle N_{L_v/K_v}(K_2(L_v)), N_{M_i^{(v)}/K_v}(K_2(M_i^{(v)})) \rangle_{1 \le i \le r_v}$$

and $Z(M_i^{(v)}) \neq \emptyset$.

Step 4: Globalize the $M_i^{(v)}$'s:

Find $N_i^{(v)}/K$ s.t. $Z(N_i^{(v)}) \neq \emptyset$ & $N_i^{(v)} \subseteq M_i^{(v)}$.

$\rightsquigarrow$ ~~Can be done by Greenberg's Approximation Th.~~

Replace the argument by another one that uses Hilbertianity prop. of K and local inverse function Th. (Z has to be smooth).

Step 5: Local-to-global principle?

$$\chi_K(Z,E)^2 \cdot x \in \underbrace{\mathrm{Ker}\left[\frac{K_2(K)}{\langle N_{L/K}(K_2(L)), N_{N_i^{(v)}/K}(K_2(N_i^{(v)}))\rangle} \to \prod_{v \in C} \frac{K_2(K_v)}{\langle N_{L_v/K_v}(K_2(L_v)), N_{M_i^{(v)}/K_v}(K_2(M_i^{(v)}))\rangle}\right]}_{= \text{Ш}_{N_2}}$$

Poitou-Tate duality:

Ш_{N_2} is dual to some $\overline{\text{Ш}^2(\hat{T})}$

for some finitely gener. torsion free Galois module $\hat{T}$.

$$\text{Ш}^2(\hat{T}) = \mathrm{Ker}\left(H^2(K,\hat{T}) \to \prod_v H^2(K_v,\hat{T})\right).$$

$\overline{A}$:= quotient of A by its maximal div. subgp.

Want to prove $\text{Ш}^2(\hat{T})$ is divisible.

$$Ш^2(\hat{T}) \longrightarrow \bigoplus_{i,v} \boxed{Ш^2(N_i^{(v)}, \hat{T})}$$

$\mathbb{Z}$.

We need to prove that the arrow satisfies some injectivity property & we need to be able to avoid some counter-ex. to Čebotarev's Th. in this context.

Need the $N_i^{(v)}$'s to be linearly disjoint pairwise and linearly disjoint to some fixed finite extension K_0/K.

Need to have some linearly disjointness assumption on the $N_i^{(v)}$'s & be able to reduce to the case where

$$\boxed{\mathrm{ires}(C) := \gcd\left(f(k'/k) \mid C(k') \neq \emptyset\right) = 1.}$$

Step 1: Reduce to Z smooth.
(devissage technique developed by Wittenberg).

Step 2: Reducing to the case $\mathrm{ires}(C) = 1$.

$\mathbb{F}_q(C)$. cd 2 — C_0^2 ✓ (KK). / C_1^1 ✓ / C_2^0 C_2. ✓

C^q_{HS}: hom. spaces under linear connected gps.

$C^q_{HS} \Leftrightarrow \mathrm{cd} \leq q+1. \longrightarrow$ Serre conj. I.

$\Updownarrow$

C_1^q

C^q_{Jc} torsors under ssc gps. $\longleftarrow$ Serre's conj. II.

$\Updownarrow$

C_2^q