Semialgebraically connected intersections of two quadrics in 5-dimensional projective space over a real closed field

joint work with Federico Scavia and Alena Pirutka

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JLCT, A. Pirutka et F. Scavia, Variétés réelles semi-algébriquement connexes non stablement rationnelles, juin 2025, https ://arxiv.org/abs/2505.21477

JLCT et A. Pirutka, Certaines fibrations en surfaces quadriques réelles, juin 2024, https ://arxiv.org/abs/2406.00463

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This talk is a sequel to Alena Pirutka's talk.
Let us just recall a few facts.
Let $\mathbb{R}$ be the reals. If a smooth projective, geometrically integral variety $X/\mathbb{R}$ is (stably rational, i.e.(stably) birational to projective space of the same dimension, then
(a) The $\mathbb{C}$-variety $X \times_{\mathbb{R}} \mathbb{C}$ is (stably) rational over $\mathbb{C}$.
(b) The set $X(\mathbb{R})$ of real points is connected.
In dimension $d = 1$ and $d = 2$ (Comessatti), the converse holds.
One wonders whether this still holds in higher dimension.

Two obvious such classes to investigate in dimension 3 and beyond :

Varieties X with a "good" fibration $X \to \mathbb{P}^1_{\mathbb{R}}$ whose generic fibre is a smooth quadric of dimension at least 2.

Smooth complete intersections of two quadrics in $\mathbb{P}^n_{\mathbb{R}}$, $n \geq 5$.

In her talk, Alena Pirutka has already recalled various birational invariants that might detect the lack of stable rationality of a smooth projective connected variety X over a field k, assuming it has a k-point :

– The Brauer group $\mathrm{Br}(X)$, more generally higher unramified cohomology.

– The Chow group of zero-cycles of degree zero $CH_0(X)$.

If X is stably rational, then over any field F/k the degree map $CH_0(X_F) \to \mathbb{Z}$ is an isomorphism : X is universally CH_0-trivial, in other terms there is an integral decomposition of the diagonal.

In her talk, A. P. discussed good, but “special” quadric fibration over $\mathbb{R}$ given by equations $x^2 + y^2 + z^2 = u.p(u)$ with $p(u) \in \mathbb{R}[u]$ positive. For many classes among these, but not all, by rather delicate computations, we could establish CH_0-triviality. For the missing ones, the question over $\mathbb{R}$ is open. However over the real Puiseux field, there are examples of such special fibrations (Benoist-Pirutka) for which stable rationality fails.

A. P. also displayed an example of bad quadric surface fibration X over $\mathbb{P}^1_{\mathbb{R}}$ with $X(\mathbb{R})$ connected which is not stably rational, because its unramified Brauer group is not reduced to $\mathrm{Br}(\mathbb{R})$.
In the present talk, we shall demonstrate that at the cost of replacing $\mathbb{R}$ by the real Puiseux field (a real closed field), starting from such an example over $\mathbb{R}$, one may produce "good", nonspecial quadric surface fibrations over $\mathbb{R}\{\{t\}\}$ which are not stably rational but whose set of "real" points is "connected".
By the same argument, we shall also produce such examples among smooth intersections of two quadrics.
This is a joint work with Alena Pirutka and Federico Scavia.

A real field is a field in which -1 is not a sum of squares. A real closed field F is a real field with no nontrivial real algebraic extension. Equivalently the field $F[t]/(t^2+1)$ is algebraically closed.

Example : the field $R = \cup_{n=1}^{\infty} \mathbb{R}((t^{1/n}))$ of real Puiseux series. We shall denote it $\mathbb{R}\{\{t\}\}$. This field is equipped with a unique order, for which $t > 0$ is smaller than any real number in $\mathbb{R}_{>0}$.

Let R be a real closed field. A semi-algebraic set in R^n is a subset belonging to the smallest family containing sets $P(x_1, \ldots, x_{n)} > 0$ (with P a polynomial) and stable under finite union, finite intersection, and taking complements.
This definition is extended to the set $X(R)$ of R-points of an algebraic variety X/R.
The closure of a semi-algebraic set is semi-algebraic.

A semi-algebraic set in $A \subset X(R)$ is called (semi-algebraically) connected if for each pair F_1, F_2 of semi-algebraic subsets closed in A with $A = F_1 \cup F_2$, either $F_1 = A$ or $F_2 = A$.
Any semi-algebraic set A in $X(R)$ is the union of finitely many semi-algebraically connected subsets $C_1, \ldots, C_s$ of A which are open and close in A.
(For $R = \mathbb{R}$, the notion coincides with the usual topological connectedness condition.)

The notion was studied by H. Delfs and M. Knebusch (1981), with later works by Coste and Roy (Springer Ergebnisse), and by C. Scheiderer.

Definition. Over a field k of char. zero, a "good" quadric surface fibration over $\mathbb{P}^1_k$ is a smooth connected threefold X equipped with a flat morphism $X \to \mathbb{P}^1_k$ all fibres of which are quadric surfaces, and all geometric singular fibres of which are irreducible.
(In other words, the finitely many geometric singular fibres are cones over a conic.)

Let us recall the "bad" quadric surface fibration $Y \to \mathbb{P}^1_{\mathbb{R}}$, whose set of real points is connected, and which are not stably rational. In $\mathbb{P}^3 \times \mathbb{A}^1_u$ with coordinates $(x, y, z, t); u$ it is given by the following equation :

$$x^2 + (1 + u^2)y^2 - u(w^2 + z^2) = 0.$$

The geometric fibres on points other than $u = 0, \infty$ are irreducible. The fibres over $u = 0$ and $u = \infty$ break up over $\mathbb{C}$ into two conjugate planes, the intersection of which is a line. The only singular points on Y are conjugate complex points on each of these two lines. One easily checks that $Y(\mathbb{R})$ is connected. One may write a desingularisation $\tilde{Y} \to Y$ with $\tilde{Y}(\mathbb{R}) = Y(\mathbb{R})$, hence connected.

(a) The map $\mathrm{Br}(\mathbb{R}(u)) \to \mathrm{Br}(\mathbb{R}(Y))$ is injective (because the determinant of the generic quadric is not a square).

(b) The nonconstant class $(-1, u) \in \mathrm{Br}(\mathbb{R}(u))$ becomes unramified on $\tilde{Y}$.

Thus $\mathrm{Br}_{nr}(\mathbb{R}(\tilde{Y}))/\mathrm{Br}(\mathbb{R}) \neq 0$ and $\tilde{Y}$ is not CH_0-trivial.

One can also produce such examples among singular intersections of two quadrics in $\mathbb{P}^5_{\mathbb{R}}$.
An easy birational transformation shows that the variety Y above is $\mathbb{R}$-birational to the singular intersection W of two quadrics in $\mathbb{P}^5_{\mathbb{R}}$ given by the system

$$q_1(x,y,z,r,u,v) = x^2 + y^2 + z^2 - zr = 0,$$

$$q_2(x,y,z,r,u,v) = u^2 + v^2 - yr = 0.$$

The only singular points on W are 4 complex points. The space $W(\mathbb{R})$ is connected (birational invariance of the number of connected components). We have $\mathrm{Br}_{nr}(\mathbb{R}(W)/\mathbb{R})/\mathrm{Br}(\mathbb{R}) \neq 0$. Hence no desingularisation of W is CH_0-trivial.

The original hope : deform these examples to good, smooth projective varieties over $\mathbb{R}$ with the same properties.

Let $p : \mathcal{X} \to U \subset \mathbb{A}^1_{\mathbb{R}}$ be a flat proper morphism to an open set U, with $O \in U(\mathbb{R})$. Assume that the morphism p is smooth outside of O, and that the special fibre $\mathcal{X}_O$ has no singular real point, and that $\mathcal{X}_O(\mathbb{R})$ is connected. Ehresmann's theorem on $\mathbb{C}^\infty$ manifolds then gives that locally around O, the fibration $\mathcal{X}(\mathbb{R}) \to U(\mathbb{R})$ is a product. In particular for any $P \in U(\mathbb{R})$ close to O, $\mathcal{X}_P(\mathbb{R})$ is connected.
Suppose we know that the special fibre $\mathcal{X}_O/\mathbb{R}$ is not stably rational thanks to some unramified cohomology obstruction. Can we find points $P \in U(\mathbb{R})$ close to O such that $\mathcal{X}_P/\mathbb{R}$ is not stably rational ?

In complex geometry, starting some 10 years ago, a technique was started by C. Voisin, developed by CT-Pirutka and later by S. Schreieder and others to disprove stable rationality of "very general" varieties of many classes of varieties which a priori are close to being rational. One uses a fibration $p : \mathcal{X} \to U \subset \mathbb{A}^1_{\mathbb{C}}$, assumes that the special, singular, fibre $\mathcal{X}_O$ is not stably rational *thanks to an unramified cohomology obstruction*, then uses specialisations of zero-cycles to deduce that the generic geometric fibre is not stably rational. The method requires that *the special, singular fibre, admits a reasonable desingularisation* (CH_0-trivial resolution of singularities). One then shows that over the points of $U(\mathbb{C})$ outside a countable union of *proper algebraic subsets* of $\mathbb{A}^1_{\mathbb{C}}$, i.e. outside a countable union of points, the fibres are not CH_0-trivial, hence not stably rational.

If one tries to mimic the argument in the real context, one only gets that the set of real points in $U(\mathbb{R})$ which one must a priori eliminate is a countable union of *proper semi-algebraic subsets* of $U(\mathbb{R})$. Typically, one would have to find a real point different from $t = 0$ in the intersection of the intervals $[-1/m, 1/m]$ for all integers $m > 0$.

Well, if we accept to pass from $\mathbb{R}$ to the Puiseux series field $\mathbb{R}\{\{t\}\}$ we do find a nonzero element, the infinitely small element t, in this intersection !

(This is not the formal argument.)

The substitute for Ehresmann's theorem.

Theorem A (on connexity)
Let $p : \mathcal{X} \to U \subset \mathbb{A}^1_{\mathbb{R}} = Spec(\mathbb{R}[t])$ *be a flat proper morphism, with* $O \in U(\mathbb{R})$ *given by* $t = 0$. *Let* $\mathcal{X}_\eta = \mathcal{X} \times_U \mathbb{R}\{\{t\}\}$. *Assume that the morphism* p *is smooth outside of* O, *and that the special fibre* $\mathcal{X}_O/\mathbb{R}$ *has no singular real point. Then the number of semi-algebraic connected components of* $\mathcal{X}_\eta(\mathbb{R}\{\{t\}\})$ *is the same as the number of connected components of* $\mathcal{X}_O/\mathbb{R}$.

In the paper, we give two proofs, none of which is simple-minded. The first one uses a Nash-triviality theorem of Coste and Shiota (1992). (Nash functions are analytic functions which are algebraic over algebraic functions.)
The other one uses Claus Scheiderer's *Real and étale cohomology* (Springer, 1994) which develops analogues of the smooth and proper base change theorems (SGA4) in the real context.

In the real context, we have to develop a version of the specialisation method as elaborated in (CT-Pirutka 2015).

The new point in this context is described in the next two slides. The hypothesis (i) is some kind of a substitute for the notion of CH_0-trivial resolution of singularities.

Proposition *Let* $p : Z \to Y$ *a proper birational* $\mathbb{R}$*-morphism of geometrically integral projective* $\mathbb{R}$*-varieties. Assume that* $Z/\mathbb{R}$ *is smooth. Let* $S \neq \emptyset$ *be the singular locus of* Y*. Let* $U \subset Y$ *be the smooth locus. Assume* $p : p^{-1}(U) \to U$ *is an isomorphism, and there exists* $b \in U(\mathbb{R})$*. Assume :*

(i) *The morphism* $S \to \mathrm{Spec}\,(\mathbb{R})$ *factorizes through* $\mathrm{Spec}\,(\mathbb{C}) \to Spec(\mathbb{R})$*. [If* S *is finite, this means* $S(\mathbb{R}) = \emptyset$*.]*

(ii) *The map*

$$\mathrm{Br}(\mathbb{R}) \to \mathrm{Ker}[\mathrm{Br}(Z) \to \mathrm{Br}(Z_{\mathbb{C}})]$$

is not surjective.

Then the difference between the generic point of Y *and the point* $b_{\mathbb{R}(Y)}$ *in* $CH_0(Y_{\mathbb{R}(Y)})$ *does not vanish.*

Proof. Let $T = p^{-1}(S)$. One compares the localization sequences for Chow groups of zero-cycles when passing over from Y to U et from Z to $p^{-1}(U)$, over the fields $\mathbb{R}(Y)$ and $\mathbb{C}(Y)$. Let $N_{\mathbb{C}/\mathbb{R}}$ denote $\mathrm{Norm}_{\mathbb{C}/\mathbb{R}}$. (i) gives that the projection map

$$\Phi : CH_0(Z_{\mathbb{R}(Y)})/N_{\mathbb{C}/\mathbb{R}}(CH_0(Z_{\mathbb{C}(Y)})) \to CH_0(Y_{\mathbb{R}(Y)})/N_{\mathbb{C}/\mathbb{R}}(CH_0(Y_{\mathbb{C}(Y)}))$$

is an *isomorphism*. This map sends the generic point of Z to the generic point of Y.

Fix $\alpha \in \mathrm{Ker}[\mathrm{Br}(Z) \to \mathrm{Br}(Z_{\mathbb{C}})]$ nonzero, with $\alpha(b) = 0$. We have $\mathrm{Br}(Z) \subset \mathrm{Br}(\mathbb{R}(Z))$. Let η be the generic point of Z. There is a pairing

$$CH_0(Z_{\mathbb{R}(Y)})/N_{\mathbb{C}/\mathbb{R}}(CH_0(Z_{\mathbb{C}(Y)})) \times \mathrm{Ker}[\mathrm{Br}(Z) \to \mathrm{Br}(Z_{\mathbb{C}})] \to \mathrm{Br}(\mathbb{R}(Y))$$

with $< \eta - b_{\mathbb{R}(Y)}, \alpha >= \alpha_{\mathbb{R}(Y)} \neq 0 \in \mathrm{Br}(\mathbb{R}(Y)) = \mathrm{Br}(\mathbb{R}(Z))$.

Via the isomorphism Φ above, one concludes that *the difference between the generic point of Y and the constant point $b_{\mathbb{R}(Y)}$ is a nontrivial class in $CH_0(Y_{\mathbb{R}(Y)})/N(CH_0(Y_{\mathbb{C}(Y)}))$. In particular the difference is not zero in $CH_0(Y_{\mathbb{R}(Y)})$.*

Theorem B (on stable rationaliy) *Let* $U \subset \mathbb{A}^1_{\mathbb{R}}$ *be an open set containing* $t = 0$*. Let* $\mathcal{X} \to U$ *be a proper map with smooth generic fibre. Assume that the special fibre* $Y = \mathcal{X}_0/\mathbb{R}$ *satisfies the conditions of the previous theorem. Then the* $\mathbb{R}\{\{t\}\}$*-variety* $\mathcal{X} \times_U \mathbb{R}\{\{t\}\}$ *is not universally* CH_0*-trivial, in particular it is not stably rational.*

Sketch of proof. If it is universally CH_0-trivial, then over some $\mathbb{R}((t^{1/n}))$ one uses specialisation of zero-cycles from $\mathcal{X} \times_U \mathbb{R}((t^{1/n}))$ to zero-cycles on $Y/\mathbb{R}$ and deduces that the generic point of the (singular) variety Y is rationally equivalent over $\mathbb{R}(Y)$ to a point coming from $Y(\mathbb{R})$. Contradiction with the above proposition.

Starting from the singular examples earlier described and putting Theorems A and B together we get :

Theorem (CT-Pirutka-Scavia 2025)

Over the field $\mathbb{R}\{\{t\}\}$ *of real Puiseux series, among each of the following classes of varieties* $X/\mathbb{R}\{\{t\}\}$

– Smooth intersections of two quadrics in $\mathbb{P}^5_{\mathbb{R}\{\{t\}\}}$.

– Good, nonspecial, quadric surface fibrations over $\mathbb{P}^1_{\mathbb{R}\{\{t\}\}}$

there exist examples such that

(i) $X(\mathbb{R}\{\{t\}\})$ *is s.a. connected,*

and

(ii) $X_{\mathbb{R}\{\{t\}\}}$ *is not universally* CH_0*-trivial, and in particular not stably rational.*

The deformation $\mathcal{X}/\mathbb{A}^1_t$ is produced in an obvious way. If $q_1 = q_2 = 0$ over $\mathbb{R}$ is the singular intersection of two quadrics constructed above, one chooses a general smooth complete intersection of two quadrics $g_1 = g_2 = 0$ and then one considers the family

$$q_1 + tg_1 = q_2 + tg_2 = 0.$$

A refined version of the technique using $H^3_{nr}(\mathbb{R}(Y)/\mathbb{R}, \mathbb{Z}/2)$ instead of the Brauer group $H^2_{nr}(\mathbb{R}(Y)/\mathbb{R}, \mathbb{Z}/2)$ leads to :

Theorem (CT-Pirutka-Scavia 2025)

Over the field $\mathbb{R}\{\{t\}\}$, among each of the following classes of varieties $X/\mathbb{R}\{\{t\}\}$

– Smooth intersections of two quadrics in $\mathbb{P}^9_{\mathbb{R}\{\{t\}\}}$.

– Good hyperquadric fibrations of relative dimension 6 over $\mathbb{P}^1_{\mathbb{R}\{\{t\}\}}$

there exist examples such that

(i) $X(\mathbb{R}\{\{t\}\})$ is s.a. connected,

and

(ii) $X_{\mathbb{R}\{\{t\}\}}$ is not universally CH_0-trivial, and in particular not stably rational.

For smooth intersection of two quadrics $X \subset \mathbb{P}^n_R$ over a real closed field, with $X(R)$ s. a. connected, we have :

For $n = 4$ (Comessatti) and $n = 6$ (Hassett, Kollár, Tschinkel 2022), X is rational. For $n = 8$, HKT also have some positive results on rationality.

There exist $X \subset \mathbb{P}^5_{\mathbb{R}}$ with $X(\mathbb{R})$ connected and X not rational (IJT obstruction, Hassett-Tschinkel, Benoist-Wittenberg, 2020).

For $n = 5$ and $n = 9$, and $R = \mathbb{R}\{\{t\}\}$, there exists such X/R which is not stably rational (above, CT-Pirutka-Scavia 2025).

[In the study of smooth complete intersection of two quadrics in $\mathbb{P}^n$, there often is a strong difference between the case n even and the case n odd.]