The Brauer-Manin obstruction for curves over global function fields

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- k is a global field, i.e., a number field or the function field of a curve over a finite field.
- X is a nice variety over a global field k.
- We are interested in the set $X(k)$ of k-rational points of X and how it sits inside the set $X(\mathbb{A}_k)_\bullet = \prod_{v \in \Omega_k} X(k_v)$ of adelic points (modified at archimedean places).
- Techniques such as descent or Brauer-Manin obstruction cut out intermediate sets

$$X(k) \subset \overline{X(k)} \subset X(\mathbb{A}_k)_\bullet^{obstruction} \subset X(\mathbb{A}_k)_\bullet$$

What is known for curves?

Conjecture [Scharaschkin, Skorobogatov, Poonen, Stoll]

Let X/k be a nice curve over a global field k. Then

$$\overline{X(k)} = X(\mathbb{A}_k)_\bullet^{\mathrm{Br}}.$$

- Known for curves with $\mathrm{Jac}(X)(k)$ finite and $\text{Ш}(\mathrm{Jac}(X)/k)_{\mathrm{div}} = 0$.
- Verified in many numerical examples with $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$.
- $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$ for most hyperelliptic curves (Bhargava-Gross-Wang).
- No examples over number fields with genus ≥ 2 and $X(k) \neq \emptyset$ for which we can prove $X(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = X(k)$

What is known in the function field case?

- In the function field case, i.e., when $k = \mathbb{F}_q(D)$ for a curve $D/\mathbb{F}_q$ much more is known.

Theorem (C.-Voloch 2023)

If X/k is a nonisotrivial curve over a global function field of genus ≥ 2, then $X(k) = X(\mathbb{A}_k)^{\mathrm{Br}}$.

- Builds on work of Poonen-Voloch (2010) which proves the theorem for X such that
  - $\mathrm{Jac}(X)$ has no nonistrovial isogeny factor, and
  - $\mathrm{Jac}(X)(k^s)[p^\infty]$ is finite.
- Other key inputs from Rössler (2013) (concerning $J(k^s)[p^\infty]$) and Abramovich-Voloch (1992) concerning Mordell-Lang over function fields.

What is a nonisotrivial curve?

Definition

Suppose $F \subset k$ is a subfield. We say X **can be defined over** F if there exists X_0/F such that $X \simeq X_0 \times_F k$.

Definition

Suppose $k = \mathbb{F}(D)$ is the function field of a curve D.

- X/k is **constant** if X can be defined over $\mathbb{F}$.
- X/k is **isotrivial** if there is some L/k such that $X \times_k L$ is constant.

Example: Suppose $f(x) \in \mathbb{F}[x]$ is separable, then

- $y^2 = f(x)$ defines a constant curve over $k = \mathbb{F}(t)$.
- $ty^2 = f(x)$ defines an isotrivial curve over $k = \mathbb{F}(t)$.

Fields of definition

From now on $k = \mathbb{F}(D)$ with $\mathbb{F}$ a finite field of characteristic p. There are purely inseparable subfields

$$k \supset k^p \supset k^{p^2} \supset \cdots \supset k^{p^n} \supset \cdots$$

with $\bigcap_{n>1} k^{p^n} = \mathbb{F}$.

Lemma 1

X/k is isotrivial if and only if it can be defined over k^{p^n} for all n.

Example: $ty^2 = f(x)$ is isomorphic to $t^{p^n} y^2 = f(x)$

Lemma 2 [Szpiro, Voloch, Abramovich-Voloch]

Suppose X is a curve of genus ≥ 2 and $Y \to X$ is a torsor under a finite abelian group whose connected component is defined over k^{p^n}. If Y can be defined over k^{p^n}, then so can X.

Frobenius and Verschiebung

There is a morphism

$$F : X \to X^{(p)}$$

called the absolute **Frobenius** given by raising coordinates to the p-th power; defining equations for $X^{(p)}$ are obtained from those of X by raising coefficients to the p-th power. Note that if X is defined over k, then $X^{(p)}$ is defined over k^p.

If A/k is an abelian variety, $F : A \to A^{(p)}$ is an isogeny of degree $p^{\dim(A)}$ and there is a complimentary isogeny, called **Verschiebung** $V : A^{(p)} \to A$ such that $V \circ F = [p]$.

Zariski dense adelic points and descent

Let $(x_v) \in X(\mathbb{A}_k) = \prod X(k_v)$ be an adelic point.

Definition

We say (x_v) is **Zariski dense** if for any closed subvariety $Z \subsetneq X$ there exists some v such that $x_v \notin Z(k_v)$.

Suppose $X \subset A$ is a curve embedded abelian variety.
Multiplication by n on A pulls back to give a $A[n]$-torsor $X' \to X$.

Defintion

We say (x_v) **survives n-descent** (with resepect to $X \subset A$) if (x_v) lifts to a twist of $X' \to X$ by an element in $\mathrm{H}^1(k, A[n])$.

Remark: If X is a curve and $(x_v) \in X(\mathbb{A}_k)^{\mathrm{Br}}$, then X survives n-descent for all n with respect to all possible A.

Proposition [C.-Voloch]

Suppose $X \subset A$ is a curve contained in an abelian variety and $(x_v) \in X(\mathbb{A}_k)$ is a Zariski dense adelic point which survives p^n-descent for all n. Then X is isotrivial.

Sketch: Let $(x'_v) \in X'(\mathbb{A}_k)$ be a lift of (x_v) to a p^n-covering

$$\begin{array}{ccccc} X' & \longrightarrow & Y & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{F^n} & A^{(p^n)} & \xrightarrow{V^n} & A \end{array}$$

- $Y \subset A^{(p^n)}$ contains a Zariski dense point with coordinates in k^{p^n}, so Y can be defined over k^{p^n}.
- $Y \to X$ is a torsor under $A^{(p)}[V^n]$ so X can also be defined over k^{p^n} by Lemma 2.
- Conclude that X is isotrivial by Lemma 1.

Question: Is this also true when $p \neq \text{char}(k)$?

Brauer-Manin for zero-dimensional subschemes of abelian varieties

Theorem

Suppose $Z \subset A$ is a zero-dimensional subscheme of an abelian variety over a global field. Then $Z(\mathbb{A}_k)_\bullet \cap A(\mathbb{A}_k)_\bullet^{\mathrm{Br}} = Z(k)$.

- Proved over number fields by Stoll (2006).
- Proved over global function fields assuming $A(k^s)[p^\infty]$ is finite by Poonen-Voloch (2010).
- We remove the assumptions on $A(k^s)[p^\infty]$ using work of Rössler (2013).

Proof of the main theorem

Theorem (C.-Voloch 2023)

If X/k is a nonisotrivial curve over a global function field of genus ≥ 2, then $X(k) = X(\mathbb{A}_k)^{\mathrm{Br}}$.

- Suppose $(x_v) \in X(\mathbb{A}_k)^{\mathrm{Br}}$.
- Then $(x_v) \in Z(\mathbb{A}_k)$ for some finite subscheme $Z \subset X$ by the proposition.
- But $Z(\mathbb{A}_k) \cap X(\mathbb{A}_k)^{\mathrm{Br}} = Z(k)$ by previous theorem, so (x_v) is k-rational.

What about the isotrivial case?

$X(k)$ can be infinite and contain Zariski dense points.

Example

Consider $k = \mathbb{F}_p(D)$ and $X = D \times_{\mathbb{F}_p} k$.

- $X(k) = \mathrm{Mor}_{\mathbb{F}_q}(D, D)$ contains the identity map which is not contained in any proper closed subvariety of D.
- $X(k)$ also contains the Frobenius morphisms $F^n : D \to D$ which are all distinct.

- It is still conjectured that $\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$, but one must take the topological closure in $X(\mathbb{A}_k)$.
- This conjecture can be reduced to the constant case (joint work in progress with Pajwani), but the constant case is still open in genus > 1.

The constant case

Theorem (C.-Voloch)

Suppose X is a constant curve over $\mathbb{F}_q(D)$.

- *If $g(D) < g(X)$, then $X(\mathbb{A}_k)^{\mathrm{Br}} = X(k) = X(\mathbb{F}_q)$.*
- *If $\mathrm{Jac}(X)$ is not an isogeny factor of $\mathrm{Jac}(D)$, then $X(\mathbb{A}_k)^{\mathrm{et-Br}} = X(k) = X(\mathbb{F}_q)$.*
- *$X(\mathbb{A}_k)^{\mathrm{et-Br}} \neq X(\mathbb{F}_q)$ iff exists $\pi_1^{et}(D) \to \pi_1^{et}(X)$ satisfying certain conditions.*

Remark: The Brauer group contains inseparable abelian descent information. The etale descent contains nonabelian descent information.

Summary

Conjecture [Scharaschkin, Skorobogatov, Poonen, Stoll]

Let X/k be a nice curve over a global field k. Then

$$\overline{X(k)} = X(\mathbb{A}_k)^{\mathrm{Br}}$$

- When k is a number field there are numerical examples with $X(k) = \emptyset$, but few general results.
- When k is a function field it is
  - known for all nonisotrivial curves with $g \geq 2$,
  - open for isotrivial curves with $g \geq 2$,
  - some general results in cases with $X(k) = X(\mathbb{F})$, but none with $X(k) \neq X(\mathbb{F})$.