Galois cohomology of reductive groups over global fields

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Thank you for inviting me to give a talk in this seminar.

Let F be a field,
$\overline{F}$ be a fixed algebraic closure of F,
$F^s \subseteq \overline{F}$ be the separable closure of F in $\overline{F}$.

If $\mathrm{char}(F) = 0$, then $F^s = \overline{F}$.
Let $\Gamma_F = \mathrm{Gal}(F^s/F)$, the absolute Galois group of F. Then $(F^s)^{\Gamma_F} = F$.

Let G be a *linear algebraic group over F*. We may regard G as $G \subseteq \mathrm{GL}_n(\overline{F})$ defined by polynomials with coefficients in F.

We denote by $G(F^s) \subseteq \mathrm{GL}_n(F^s)$ and $G(F) \subseteq \mathrm{GL}_n(F)$ the corresponding groups of points. Then Γ_F acts of $G(F^s)$ via the action on the matrix entries

$$(\gamma, g) \mapsto {}^{\gamma}g \quad \text{for } \gamma \in \Gamma_F,\ g \in G(F^s).$$

We have

$$G(F^s)^{\Gamma_F} = G(F).$$

Galois cohomology

$Z^1(F,G)$ is the set of locally constant maps $c\colon \Gamma_F \to G(F^s)$ satisfying the *cocycle condition*

$$c(\gamma\delta) = c(\gamma)\cdot {}^{\gamma}c(\delta) \quad \text{for all } \gamma,\delta \in \Gamma_F.$$

The group $G(F^s)$ acts on $Z^1(F,G)$ on the right by *twisted conjugation*

$$(c * g)(\gamma) = g^{-1}\cdot c(\gamma)\cdot {}^{\gamma}g \quad \text{for } c \in Z^1(F,G),\ g \in G(F^s),\ \gamma \in \Gamma_F.$$

By definition,

$$H^1(F,G) = Z^1(F,G)/G(F^s).$$

A priori $H^1(F,G)$ is just a pointed set (not a group),
with a distinguished point 1, the class of the cocycle 1.

If G is commutative, then $H^1(F,G)$ is naturally an abelian group,
and one can define abelian groups $H^i(F,G)$ for all $i \geq 0$.

Example

Assume that G is a *constant finite* F-group, that is, the group $G(F^s)$ is finite and Γ_F acts on $G(F^s)$ *trivially*; in other words, $G(F^s) = G(F)$. Then

$$Z^1(F, G) = \mathrm{Hom}\big(\Gamma_F, G(F^s)\big)$$

and

$$H^1(F, G) = \mathrm{Hom}\big(\Gamma_F, G(F^s)\big)/\text{conjugation.}$$

For general G, the definition of $H^1(F, G)$ it not intuitive, and we cannot compute the Galois cohomology directly from the definition, in particular because we do not know the Galois group Γ_F.

Applications of Galois cohomology

Let X be a *quasi-projective* algebraic variety with additional structure, defined over F (for example, an algebraic F-group or a homogeneous space). Assume that $G = \mathrm{Aut}(X)$. We wish to classify the twisted F^s/F-forms of X, that is, the isomorphism classes of F-varieties with similar structure X' such that

$$X' \times_F F^s \simeq X \times_F F^s.$$

They are classified by $H^1(F, G)$.

Similarly, let V be a finite dimensional vector space over F, and let

$$t \in (V^*)^{\otimes m} \otimes V^{\otimes n}$$

be a tensor (for example, a bilinear form or a structure of Lie algebra). Write

$$G = \mathrm{Stab}(t) \in \mathrm{GL}(V).$$

Then the twisted forms of the pair (V, t) are classified by $H^1(F, G)$.

We see that in all classification problems over a nonclosed field in algebraic geometry and linear algebra, one needs Galois cohomology.

Global and local fields

The theory of Galois cohomology $H^1(F,G)$ does depend on the field F. For example, when $F=\mathbb{F}_q$ is a *finite field*, we have $H^1(F,G)=1$ for all *connected* groups G (Lang's theorem).

I will discuss $H^1(F,G)$ in the case when F is a *local field* or a *global field*.

A *global field of characteristic* 0 is a *number field*: a finite extension of the field of rational numbers $\mathbb{Q}$.

A *global field* F *of characteristic* $p>0$ is a *global function field*: the field of rational functions on an algebraic curve over a finite field $\mathbb{F}_q$ of cardinality $q=p^l$ for some natural l. In other words, F is a finite extension of the field $\mathbb{F}_q(x)$ of rational functions in one variable over a finite field $\mathbb{F}_q$.

We consider *places* v of our global field F, that is, absolute values up to equivalence. Then F_v denotes the completion of F with respect to v. These completions are called *local fields*.

Algebraic fundamental group of a reductive group

Let G be a *connected reductive* algebraic group over a field F.
Example. $G = \mathrm{GO}_{n,F} = \{g \in \mathrm{GL}(n, \overline{F}) \mid g \cdot g^T = \lambda I_n,\ \lambda \in \overline{F}^\times\}$.

Let G^{sc} be the *universal cover* of the *commutator subgroup* $[G,G]$ of G.
In our example, $[G,G] = \mathrm{SO}_{n,F}$ and $G^{\mathrm{sc}} = \mathrm{Spin}_{n,F}$. Consider

$$\rho\colon\ G^{\mathrm{sc}} \twoheadrightarrow [G,G] \hookrightarrow G.$$

Let $T \subseteq G$ be a maximal torus. Set $T^{\mathrm{sc}} = \rho^{-1}(T) \subseteq G^{\mathrm{sc}}$ and consider

$$\rho\colon T^{\mathrm{sc}} \to T, \qquad \rho_*\colon \mathsf{X}_*(T^{\mathrm{sc}}) \to \mathsf{X}_*(T),$$

where $\mathsf{X}_*(T) = \mathrm{Hom}(\mathbf{G}_{m,\overline{F}}, T_{\overline{F}})$ denotes the cocharacter group of T.

We set

$$\pi_1(G) = \mathsf{X}_*(T)/\rho_*\mathsf{X}_*(T^{\mathrm{sc}}).$$

The Galois group Γ_F naturally acts on $\pi_1(G)$, and the obtained Γ_F-module does not depend on the choice of T.

In characteristic 0, our *algebraic fundamental group* $\pi_1(G)$ of B.98 can be non-formally regarded as the topological fundamental group $\pi_1^{\mathrm{top}}(G(\mathbb{C}))$ defined algebraically.

Algebraic fundamental group (cont.)

Examples.

- G is a simply connected semisimple group, say, SL_n. Then $\pi_1(G) = 0$.
- $G = \mathrm{PGL}_n = \mathrm{SL}_n/\mu_n$. Then $\pi_1(G) = \mathbb{Z}/n\mathbb{Z}$, whereas $\pi^{\text{ét}}(G) = \mu_n$.
- $G = \mathbf{G}_{m,F}$. Then $\pi_1(G) = \mathbb{Z}$, whereas $\pi^{\text{ét}}(G) = \varprojlim \mu_n = \widehat{\mathbb{Z}}(1)$.
- E/F a separable quadratic extension with Galois group $\Gamma_{E/F} = \{1, \gamma\}$ of order 2. Consider the 1-dimensional torus

$$G = R^1_{E/F}\mathbf{G}_{\mathrm{m}} := \ker\left[N_{E/F}\colon R_{E/F}\mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m},F}\right]$$

with the group of F-points $\ker[E^\times \to F^\times]$. Then $\pi_1(G) \simeq \mathbb{Z}$ (because G is a 1-dimensional torus), $\gamma \in \Gamma_{E/F}$ acts on $\pi_1(G) \simeq \mathbb{Z}$ by multiplication by -1, and Γ_F acts on $\pi_1(G)$ via $\Gamma_{E/F}$.

The abelianization map

Consider the complex of tori $\underset{-1}{T^{\mathrm{sc}}} \xrightarrow{\rho} \underset{0}{T}$ in degrees -1 and 0.

We define (using cocycles) the abelian group

$$H^1_{\mathrm{ab}}(F,G) = \mathbb{H}^1(F, T^{\mathrm{sc}} \to T) := \mathbb{H}^1\big(\Gamma_F, T^{\mathrm{sc}}(F^s) \to T(F^s)\big)$$

where the hypercohomology $\mathbb{H}^1$ is a kind of mixture of $H^1(F,T)$ and $H^2(F,T^{\mathrm{sc}})$. Our $H^1_{\mathrm{ab}}(F,G)$ depends only on the Γ_F-module $\pi_1(G)$.

Moreover, we define the *abelianization map*

$$\mathrm{ab}\colon H^1(F,G) \to H^1_{\mathrm{ab}}(F,G),$$

which fits into the exact sequence

$$H^1(F,G^{\mathrm{sc}}) \xrightarrow{\rho_*} H^1(F,G) \xrightarrow{\mathrm{ab}} H^1_{\mathrm{ab}}(F,G).$$

For a local or global field F, the map ab is *surjective*.

Abelianization map: construction

I recall the construction of the abelianization map from B.98.
We consider the complex of nonabelian groups

$$G^{\mathrm{sc}} \to G$$

with G^{sc} in degree -1. This complex is a *crossed module:* the group G naturally acts on G^{sc}, and this action is compatible with the actions of G on G and of G^{sc} on G^{sc} by conjugation. Using this structure of a crossed module, the speaker defined in B.98 the first Galois hypercohomology set

$$\mathbb{H}^1(F, G^{\mathrm{sc}} \to G).$$

The inclusion

$$(T^{\mathrm{sc}} \to T) \hookrightarrow (G^{\mathrm{sc}} \to G)$$

is a *quasi-isomorphism:* it induces isomorphisms on the kernels and the cokernels. Thus the induced map

$$\mathbb{H}^1(F, T^{\mathrm{sc}} \to T) \to \mathbb{H}^1(F, G^{\mathrm{sc}} \to G)$$

is a bijection.

Abelianization map: construction (cont.)

On the other hand, the inclusion of crossed modules

$$(1 \to G) \hookrightarrow (G^{\mathrm{sc}} \to G)$$

induces a natural map

$$H^1(F, G) = \mathbb{H}^1(F, 1 \to G) \to \mathbb{H}^1(F, G^{\mathrm{sc}} \to G).$$

We obtain our abelianization map:

$$\mathrm{ab}\colon\ H^1(F, G) \to \mathbb{H}^1(F, G^{\mathrm{sc}} \to G) \xrightarrow{\sim} \mathbb{H}^1(F, T^{\mathrm{sc}} \to T) =: H^1_{\mathrm{ab}}(F, G).$$

Cohomology over a non-archimedean local field

Write $M = \pi_1(G)$, and let M_{Γ_F} denote the *group of coinvariants* of Γ_F in M:

$$M_{\Gamma_F} = M \,/\, \langle {}^{\gamma}m - m \mid \gamma \in \Gamma_F,\, m \in M \rangle.$$

Write $M_{\Gamma_F,\mathrm{Tors}} = (M_{\Gamma_F})_{\mathrm{Tors}}$ for the torsion subgroup of M_{Γ_F}.

Theorem (B.98, González-Avilés 12, goes back to Kottwitz 86)

For a non-archimedean local field F, the map

$$\mathrm{ab}\colon H^1(F,G) \to H^1_{\mathrm{ab}}(F,G)$$

is bijective, and

$$H^1(F,G) \cong H^1_{\mathrm{ab}}(F,G) \cong M_{\Gamma_F,\mathrm{Tors}}.$$

Questions?

Over non-archimedean local fields: Trivial examples

- G is a *simply connected* semisimple F-group. Then $M \coloneqq \pi_1(G) = 0$, whence

  $$H^1(F, G) = M_{\Gamma,\mathrm{Tors}} = 0$$

  (theorem of Kneser and of Bruhat and Tits). We don't give a new proof of Kneser's theorem; we *use* it.

- $G = \mathbf{G}_{\mathrm{m},F}$, the 1-dimensional split F-torus. Then $M = \mathbb{Z}$, Γ_F acts on $M = \mathbb{Z}$ trivially,

  $$M_{\Gamma_F} = \mathbb{Z}, \qquad M_{\Gamma_F,\mathrm{Tors}} = 0, \qquad H^1(F, \mathbf{G}_{\mathrm{m},F}) = 1$$

  (well-known: Hilbert's Theorem 90).

A less trivial example

- E/F a separable quadratic extension, $G = R^1_{E/F}\mathbf{G}_{\mathrm{m}}$ is the 1-dimensional F-torus with

$$G(F) = \ker\left[N_{E/F}\colon E^\times \to F^\times\right],$$

$M \simeq \mathbb{Z}$, Γ_F acts on M via $\Gamma := \Gamma_{E/F} = \{1, \gamma\}$, where γ acts on $M \simeq \mathbb{Z}$ by ${}^\gamma m = -m$. An easy calculation shows that

$$M_{\Gamma_F,\mathrm{Tors}} = M_{\Gamma_F} \cong \mathbb{Z}/2\mathbb{Z}.$$

(indeed, $m - {}^\gamma m = m - (-m) = 2m$). Thus $H^1(F, G) \cong \mathbb{Z}/2\mathbb{Z}$. This result is well-known. Indeed, we have

$$H^1(F, G) = H^1(E/F, G) \cong F^\times/N_{E/F}(E^\times) \cong \Gamma_{E/F} \cong \mathbb{Z}/2\mathbb{Z}.$$

Over global field: what is known?

Let F be a global field,
G be a connected reductive group over F,
$\mathcal{V}_F$ the set of places of F.
For any place $v \in \mathcal{V}_F$ we have the *localization map*

$$\mathrm{loc}_v \colon H^1(F, G) \to H^1(F_v, G).$$

Thus we obtain a map

$$\mathrm{loc} \colon H^1(F, G) \to \prod_{v \in \mathcal{V}_F} H^1(F_v, G).$$

This map actually takes values in

$$\bigoplus_{v \in \mathcal{V}_F} H^1(F_v, G) := \left\{ (\xi_v) \in \prod_{v \in \mathcal{V}_F} H^1(F_v, G) \;\middle|\; \xi_v = 1 \text{ for almost all } v. \right\}.$$

What is known (cont.)

There is an exact sequence of Kottwitz 86

$$\text{(K)} \quad 1 \to \text{Ш}(F,G) \to H^1(F,G) \xrightarrow{\text{loc}} \bigoplus_{v \in \mathcal{V}_F} H^1(F_v,G) \xrightarrow{\Sigma} M_{\Gamma_F,\text{Tors}}$$

where the map Σ is easy to describe. Thus we know the *image* $\mathrm{im}\,\mathrm{loc}$.

On the other hand, the Tate-Shafarevich *kernel* $\text{Ш}(F,G) = \ker\,\mathrm{loc}$ has a canonical structure of a *finite abelian group* (Sansuc 81), and Kottwitz 86 computed this group. It can be computed in terms of $M = \pi_1(G)$ (B.98). This group $\text{Ш}(F,G)$ acts *simply transitively* on each non-empty fiber of the map loc.

Thus we know all groups and sets in the exact sequence (K), except for $H^1(F,G)$. The novel part of my talk is a *closed formula for* $H^1(F,G)$.

Global field

Let F be a global field, $M = \pi_1(G)$. Let E/F be a *finite* Galois extension in F^s such that $\mathrm{Gal}(F^s/E)$ acts on M trivially.
Set $\Gamma = \Gamma_{E/F}$; then Γ acts on M.

Let $\mathcal{V}_E$ denote the set of places of E. The Galois group $\Gamma = \Gamma_{E/F}$ acts on E, and so it naturally acts on $\mathcal{V}_E$. Consider the surjective map

$$p_{E/F} \colon \mathcal{V}_E \to \mathcal{V}_F,\ w \mapsto w|_F.$$

For $v \in \mathcal{V}_F$, set $\mathcal{V}_E(v) = p_{E/F}^{-1}(v) \subset \mathcal{V}_E$. In other words, $\mathcal{V}_E(v)$ is the set of all places (absolute values) w of E extending the place v of F. Then Γ acts on each of the finite sets $\mathcal{V}_E(v)$ transitively, that is, each $\mathcal{V}_E(v)$ is an orbit of Γ. We have

$$\mathcal{V}_E = \bigcup_{v \in \mathcal{V}_F} \mathcal{V}_E(v).$$

Cohomology over a global function field

Following Tate 66, we consider the group of *finite* formal linear combinations

$$M[\mathcal{V}_E] = \Big\{ \sum_{w\in\mathcal{V}_E} m_w \cdot w \;\Big|\; m_w \in M \Big\}$$

and the subgroup

$$M[\mathcal{V}_E]_0 = \Big\{ \sum m_w \cdot w \in M[\mathcal{V}_E] \;\Big|\; \sum m_w = 0 \Big\}$$

of such sums with zero sum of the coefficients. The finite group $\Gamma = \Gamma_{E/F}$ naturally acts on $M[\mathcal{V}_E]$ and on $M[\mathcal{V}_E]_0$.

Theorem (B-Kaletha 23)

For a global function field F, the map $\mathrm{ab}\colon H^1(F,G) \to H^1_{\mathrm{ab}}(F,G)$ is bijective, and

$$H^1(F,G) \cong H^1_{\mathrm{ab}}(F,G) \cong \big(M[\mathcal{V}_E]_0\big)_{\Gamma,\mathrm{Tors}}.$$

Questions?

Over global function field: Exact sequence

We have a short exact sequence of Γ-modules

$$0 \to M[\mathcal{V}_E]_0 \xrightarrow{i} M[\mathcal{V}_E] \xrightarrow{\Sigma} M \to 0.$$

It gives rise to the following exact sequence:

$$\bigoplus_{v \in \mathcal{V}_F} H_1(\Gamma_w, M) \xrightarrow{\Sigma_*} H_1(\Gamma, M) \xrightarrow{\delta} \big(M[\mathcal{V}_E]_0\big)_{\Gamma,\mathrm{Tors}}$$

$$\xrightarrow{i_*} \bigoplus_{v \in \mathcal{V}_F} M_{\Gamma_w,\mathrm{Tors}} \xrightarrow{\Sigma_*} M_{\Gamma,\mathrm{Tors}},$$

where for each $v \in \mathcal{V}_F$ we choose a place $w \in \mathcal{V}_E$ over v, and we denote by Γ_w the stabilizer of w in Γ.

Localization

Let F be a global function field, and let $v_1 \in \mathcal{V}_F$ be a place of F. We have *closed formulas* for $H^1(F,G)$ and $H^1(F_{v_1},G)$. I describe the localization map $\mathrm{loc}_{v_1}\colon H^1(F,G) \to H^1(F_{v_1},G)$.

I repeat: for a finite Galois extension E/F as above, we consider the surjective map

$$p_{E/F}\colon \mathcal{V}_E \to \mathcal{V}_F,\ w \mapsto w|_F.$$

For $v \in \mathcal{V}_F$, set $\mathcal{V}_E(v) = p_{E/F}^{-1}(v) \subset \mathcal{V}_E$. Then Γ acts on each of the finite sets $\mathcal{V}_E(v)$ transitively, and $\mathcal{V}_E = \bigcup_{v\in\mathcal{V}_F} \mathcal{V}_E(v)$.

Consider

$$M[\mathcal{V}_E(v)] = \Big\{ \sum m_w \cdot w \mid w \in \mathcal{V}_E(v),\, m_w \in M \Big\}.$$

Since $\mathcal{V}_E = \bigcup_{v \in \mathcal{V}_F} \mathcal{V}_E(v)$, we have $M[\mathcal{V}_E] = \bigoplus_{v \in \mathcal{V}_F} M[\mathcal{V}_E(v)]$. Consider the projection map

$$\lambda_v \colon M[\mathcal{V}_E]_0 \hookrightarrow M[\mathcal{V}_E] \twoheadrightarrow M[\mathcal{V}_E(v)]$$

and the induced map

$$l_v \colon \big(M[\mathcal{V}_E]_0\big)_{\Gamma,\mathrm{Tors}} \to \big(M[\mathcal{V}_E(v)]\big)_{\Gamma,\mathrm{Tors}} \cong M_{\Gamma_w,\mathrm{Tors}}$$

where the last isomorphism comes from the fact that Γ acts on $\mathcal{V}_E(v)$ transitively, and therefore the Γ-module $M[\mathcal{V}_E(v)]$ is *induced* by the Γ_w-module M.

This is our localization map. The map l_v is defined also for a *number field* F, though then it is not always the localization map.

H^1 over $\mathbb{R}$

Let $F = \mathbb{R}$, and let G be a connected reductive group over $\mathbb{R}$. I know how to compute $H^1(\mathbb{R}, G)$ (four papers and a computer program) in terms of combinatorial data describing G, in particular, the *root system*; however, this is a topic for a separate talk.

The abelianization map $\mathrm{ab}\colon H^1(\mathbb{R}, G) \to H^1_{\mathrm{ab}}(\mathbb{R}, G)$ is surjective. We have

$$H^1_{\mathrm{ab}}(\mathbb{R}, G) \cong \widehat{H}^{-1}(\Gamma, M) \quad \text{(B-Timashev 23)}$$

where $M = \pi_1(G), \quad \Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$, and $\widehat{H}^{-1}$ denotes Tate cohomology. This follows immediately from the Tate-Nakayama theorem for the Galois extension $\mathbb{C}/\mathbb{R}$.

The group $\widehat{H}^{-1}(\Gamma, M)$ naturally embeds into $M_{\Gamma,\mathrm{Tors}}$, and we obtain a map

$$\alpha_{\mathbb{R}}\colon H^1(\mathbb{R}, G) \xrightarrow{\mathrm{ab}} H^1_{\mathrm{ab}}(\mathbb{R}, G) \cong \widehat{H}^{-1}(\Gamma_{\mathbb{C}/\mathbb{R}}, M) \hookrightarrow M_{\Gamma_{\mathbb{C}/\mathbb{R}},\mathrm{Tors}}.$$

H^1 over $\mathbb{Q}$

Let $F = \mathbb{Q}$ and G be a connected reductive $\mathbb{Q}$-group. Write $M = \pi_1(G)$. Let E/F be as above, that is, Γ_F acts on M via $\Gamma_{E/F}$, and assume that *E has no real places.* Write $\Gamma = \Gamma_{E/F}$. Let w be a place of E over the place ∞ of $\mathbb{Q}$; then the stabilizer $\Gamma_w = \mathrm{Gal}(E_w/F_\infty) = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

Consider the maps

$$(*) \qquad (M[\mathcal{V}_E]_0)_{\Gamma,\mathrm{Tors}} \xrightarrow{l_\infty} M_{\Gamma_w,\mathrm{Tors}} \xleftarrow{\alpha_\mathbb{R}} H^1(\mathbb{R}, G).$$

Theorem (B-Kaletha 23)

$H^1(\mathbb{Q}, G)$ is in a canonical bijection with the fibered product of the maps in $(*)$. In other words, there is a bijection

$$H^1(\mathbb{Q}, G) \xrightarrow{\sim}$$
$$\left\{(x, \xi_\mathbb{R}) \;\middle|\; x \in (M[\mathcal{V}_E]_0)_{\Gamma,\mathrm{Tors}},\ \xi_\mathbb{R} \in H^1(\mathbb{R}, G),\ l_\infty(x) = \alpha_\mathbb{R}(\xi_\mathbb{R})\right\}.$$

Questions?

H^1 over a number field

Let G be a reductive group over an arbitrary number field F. Let E/F be as above. Consider the maps

$$(**) \quad (M[\mathcal{V}_E]_0)_{\Gamma_{E/F},\mathrm{Tors}} \xrightarrow{\prod_\infty l_v} \prod_\infty M_{\Gamma_w,\mathrm{Tors}} \xleftarrow{\prod_\infty \alpha_v} \prod_\infty H^1(F_v, G)$$

where $\prod_\infty$ is taken over $v \in \mathcal{V}_{F,\infty}$ (the set of archimedean places of F), and for any $v \in \mathcal{V}_{F,\infty}$ we choose $w \in \mathcal{V}_{E,\infty}$ over v.

Theorem (B-Kaletha 23)

$H^1(F, G)$ is in a canonical bijection with the fibered product of the maps in $(**)$.

Questions?

This theorem describes $H^1(F, G)$ in terms of the Γ_F-module M and the real Galois cohomology sets $H^1(F_v, G)$ for $v \in \mathcal{V}_{F,\infty}$.

A trivial example over a number field

F is a number field, G is *simply connected* semisimple group. Then $M = \pi_1(G) = 0$, and

$$\big(M[V_E]_0\big)_{\Gamma_{E/F},\mathrm{Tors}} = 0, \qquad H^1(F,G) \cong \prod_{\infty} H^1(F_v,G),$$

which is the celebrated Hasse principle of Kneser, Harder, and Chernousov. We don't give a new proof of this result; we *use* it.

$H^2(F,T)$

We also have a formula for $H^2(F,T)$ when T is an F-torus, where F is a local or global field. Write $M = \pi_1(T) = \mathsf{X}_*(T)$.

Theorem (B.98 when $\mathrm{char}(F) = 0$)

If F is a non-archimedean local field, then

$$H^2(F,T) \cong M_{\Gamma_F} \otimes \mathbb{Q}/\mathbb{Z}.$$

Theorem (B-Kaletha 23)

If F is a global function field, then

$$H^2(F,T) \cong \left(M[\mathcal{V}_E]_0\right)_{\Gamma_{E/F}} \otimes \mathbb{Q}/\mathbb{Z}$$

for a finite Galois extension E/F such that Γ_F acts on M via $\Gamma_{E/F}$.

When F is a number field, $H^2(F,T)$ is a certain fiber product.

H^1: an example over $\mathbb{Q}$.

Let $F = \mathbb{Q}$, $E = \mathbb{Q}(\sqrt{13}, \sqrt{17})$, $T = R^1_{E/\mathbb{Q}}\mathbf{G}_m$. Then $\Gamma := \Gamma_{E/\mathbb{Q}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and a calculation shows that all decomposition groups for $E/\mathbb{Q}$ are *cyclic*. It follows that $H^1(F_v, T)$ is killed by 2 for all places $v \in \mathcal{V}_\mathbb{Q}$. We have the Kottwitz exact sequence

$$0 \to Ш^1(\mathbb{Q}, T) \to H^1(\mathbb{Q}, T) \to \bigoplus_v H^1(F_v, T) \to \mathbb{Z}/2\mathbb{Z} \to 0$$

from which we see that $H^1(F, T)/Ш^1(F, T)$ is killed by 2. Moreover, Sansuc 81 showed that $Ш^1(\mathbb{Q}, T) \simeq \mathbb{Z}/2\mathbb{Z}$.

On the other hand, since $|\Gamma| = 4$, we see that the group $H^1(\mathbb{Q}, T) = H^1(\Gamma, T(E))$ is killed by 4.

Question. Is there an element of order 4 in $H^1(\mathbb{Q}, T)$?

H^1: an example over $\mathbb{Q}$ (cont.)

We have

$$H^1(\mathbb{Q}, T) \cong \big(M[V_E]_0\big)_{\Gamma,\mathrm{Tors}} = \widehat{H}^{-1}(\Gamma, M[V_E]_0).$$

Using computer, one can show that

$$\widehat{H}^{-1}(\Gamma, M[V_E]_0) \simeq \mathbb{Z}/4\mathbb{Z} \oplus A_2$$

where A_2 is an infinite abelian group killed by 2. Thus the answer is Yes, $H^1(\mathbb{Q}, T)$ does contain an element of order 4.

Questions?

H^1: an example over $\mathbb{Q}$ (details)

We compute $\widehat{H}^{-1}(\Gamma, M[V_E]_0)$ using computer as follows.
We construct a certain finite subset $X_E \subset V_E$, $|X_E| = 10$, such that there is an isomorphism of Γ-modules

$$M[V_E]_0 \simeq M[X_E]_0 \oplus M[Y_E]$$

where $Y_E = V_E \smallsetminus X_E$. We obtain an isomorphism

$$\widehat{H}^{-1}\big(\Gamma, M[V_E]_0\big) \simeq \widehat{H}^{-1}\big(\Gamma, M[X_E]_0\big) \oplus \widehat{H}^{-1}\big(\Gamma, M[Y_E]\big).$$

Since the stabilizer of each element of Y_E is of order 1 or 2, we see that $\widehat{H}^{-1}\big(\Gamma, M[Y_E]\big)$ is killed by 2.

On the other hand, I computed $\widehat{H}^{-1}(\Gamma, M[V_E]_0)$ using computer and got $\mathbb{Z}/4\mathbb{Z}$.

Idea of proof: reduction to H^1_{ab}

I want to compute $H^1(F,G)$ for a connected reductive group G over a number field F. By B.98, $H^1(F,G)$ fits into a Cartesian diagram

$$\begin{array}{ccc} H^1(F,G) & \xrightarrow{\mathrm{ab}} & H^1_{\mathrm{ab}}(F,G) \\ \downarrow{\scriptstyle \mathrm{loc}_\infty} & & \downarrow{\scriptstyle \mathrm{loc}_\infty} \\ \prod_\infty H^1(F_v,G) & \xrightarrow{\prod_\infty \mathrm{ab}_v} & \prod_\infty H^1_{\mathrm{ab}}(F_v,G) \end{array}$$

where $\prod_\infty$ means $\prod_{v\in\mathcal{V}_\infty(F)}$, the product over the infinite places of F. Here "Cartesian" means that $H^1(F,G)$ is the fiber product.

Thus computing $H^1(F,G)$ reduces to computing $H^1_{\mathrm{ab}}(F,G)$ and computing $H^1(F_v,G)$ for *real* places v of F.

Computing H^1_{ab}

For a connected reductive group G over a global field F, I wish to compute

$$H^1_{\mathrm{ab}}(F,G) \coloneqq \mathbb{H}^1(F, T^{\mathrm{sc}} \to T).$$

By a definition,

$$\mathbb{H}^1(F, T^{\mathrm{sc}} \to T) = \varinjlim_{K} \left(\mathbb{H}^1(K/F, T^{\mathrm{sc}} \to T), \mathrm{Inf}_{K'/K}\right)$$

where K runs over *finite* Galois extensions of F in F^s containing E, and for $K' \supset K$,

$$\mathrm{Inf}_{K'/K} \colon \ \mathbb{H}^1(K/F, T^{\mathrm{sc}} \to T) \to \mathbb{H}^1(K'/F, T^{\mathrm{sc}} \to T)$$

is the inflation map.

Extending a result of Tate 66 for one torus to a *complex of tori*, for any such K we obtain an isomorphism

$$\widehat{H}^{-1}(K/F, M[\mathcal{V}_K]_0) \xrightarrow{\sim} \mathbb{H}^1(K/F, T^{\mathrm{sc}} \to T).$$

where $\widehat{H}^{-1}$ denotes the Tate cohomology.

Computing H^1_{ab} (cont.)

We define a homomorphism

$$?_{K'/K}\colon \widehat{H}^{-1}(K/F, M[\mathcal{V}_K]_0) \to \widehat{H}^{-1}(K'/F, M[\mathcal{V}_{K'}]_0)$$

by the commutative diagram

$$\begin{array}{ccc}
\widehat{H}^{-1}(K/F, M[\mathcal{V}_K]_0) & \xrightarrow[\text{Tate 66}]{\sim} & \mathbb{H}^1(K/F, T^{\mathrm{sc}} \to T) \\
\downarrow {\scriptstyle ?_{K'/K}} & & \downarrow {\scriptstyle \mathrm{Inf}_{K'/K}} \\
\widehat{H}^{-1}(K'/F, M[\mathcal{V}_{K'}]_0) & \xrightarrow[\text{Tate 66}]{\sim} & \mathbb{H}^1(K'/F, T^{\mathrm{sc}} \to T)
\end{array}$$

This homomorphism $?_{K'/K}$ is *not inflation:* there is no inflation in Tate cohomology $\widehat{H}^n$ for $n \le 0$. To compute $H^1_{\mathrm{ab}}(F, G)$, it remains to guess the map $?_{K'/K}$ (which is not that hard), to prove that the diagram indeed commutes with this $?_{K'/K}$ (which was hard for me), and to compute the limit

$$\varinjlim_{K} \left(\widehat{H}^{-1}(K/F, M[\mathcal{V}_K]_0),\ ?_{K'/K}\right).$$

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Thank you!